

Manuscript version: Author's Accepted Manuscript

The version presented in WRAP is the author's accepted manuscript and may differ from the published version or Version of Record.

Persistent WRAP URL:

<http://wrap.warwick.ac.uk/141713>

How to cite:

Please refer to published version for the most recent bibliographic citation information. If a published version is known of, the repository item page linked to above, will contain details on accessing it.

Copyright and reuse:

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions.

© 2020 Elsevier. Licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International <http://creativecommons.org/licenses/by-nc-nd/4.0/>.



Publisher's statement:

Please refer to the repository item page, publisher's statement section, for further information.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk.

Three-coloring triangle-free graphs on surfaces

IV. Bounding face sizes of 4-critical graphs*

Zdeněk Dvořák[†] Daniel Král'[‡] Robin Thomas[§]

Apr 27, 2019

Abstract

Let G be a 4-critical graph with t triangles, embedded in a surface of genus g . Let c be the number of 4-cycles in G that do not bound a 2-cell face. We prove that

$$\sum_{f \text{ face of } G} (|f| - 4) \leq \kappa(g + t + c - 1)$$

for a fixed constant κ , thus generalizing and strengthening several known results. As a corollary, we prove that every triangle-free graph G embedded in a surface of genus g contains a set of $O(g)$ vertices such that $G - X$ is 3-colorable.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [8].

Theorem 1.1. *Every triangle-free planar graph is 3-colorable.*

Thomassen [15, 16, 18] found three reasonably simple proofs of this claim. Recently, two of us, in joint work with Kawarabayashi [2] were able to design a

*18 January 2015.

[†]Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague, project P202/12/G061 of Czech Science Foundation.

[‡]Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic, and Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK. E-mail: dkral@fi.muni.cz

[§]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: thomas@math.gatech.edu. Partially supported by NSF Grants No. DMS-0739366 and DMS-1202640.

linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. Kostochka and Yancey [12] gave a completely different proof as a consequence of their results on critical graphs. The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical graphs that can be embedded in Σ (a graph is 4-critical if it is not 3-colorable, but all its proper subgraphs are 3-colorable).

Gimbel and Thomassen [7] obtained the following elegant characterization of 4-critical triangle-free graphs embedded in the projective plane. A graph embedded in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

Theorem 1.2. *A triangle-free graph embedded in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

However, on other surfaces the structure of triangle-free 4-critical graphs appears more complicated. Thus we aim not for a complete characterization, but instead for a quantitative bound that can be used in applications—roughly, that each 4-critical triangle-free graph embedded in a fixed surface has only a bounded number of faces of length greater than 4. One difficulty that needs to be dealt with is that the claim as stated is false: there exist 4-critical plane graphs with arbitrarily many faces of length 5 and with exactly 4 triangles, see [14] for a construction. By replacing an edge in each of the triangles by a suitably chosen triangle-free graph, one can obtain a 4-critical triangle-free graph embedded in a surface of bounded genus and with arbitrarily many 5-faces. However, the examples constructed in this way contain a large number of non-contractible 4-cycles. This turns out to be the case in general, as shown by the main result of this paper, the following.

Theorem 1.3. *There exists a constant κ with the following property. Let G be a graph embedded in a surface of Euler genus g . Let t be the number of triangles in G and let c be the number of 4-cycles in G that do not bound a 2-cell face. If G is 4-critical, then*

$$\sum_{f \text{ face of } G} (|f| - 4) \leq \kappa(g + t + c - 1).$$

The first step towards this result was obtained in the previous paper of this series [5], where we showed that Theorem 1.3 holds under the additional assumption that the girth of G is at least five (this special case strengthens a theorem of Thomassen [17]). The inclusion of triangles and non-facial 4-cycles in the statement is merely a technicality, as (≤ 4) -cycles can be eliminated by replacing their edges by suitable graphs of girth 5 at the expense of increasing the genus. Hence, the main contribution of this paper is dealing with 4-faces.

Theorem 1.3 is the cornerstone of this series. We will use it in future papers to deduce the following results. First, in [6] (whose preliminary version appeared

as [3]) we will rely on the fact that the bound in Theorem 1.3 is linear in t to answer affirmatively a question of Havel [9] whether every planar graph with triangles sufficiently far apart is 3-colorable. Second, we will use Theorem 1.3 and another result about 3-coloring graphs with most faces bounded by 4-cycles to design a polynomial-time algorithm to test whether an input triangle-free graph embedded in a fixed surface is 3-colorable. That settles a problem of Gimbel and Thomassen [7] and completes one of two missing steps in a research program initiated by Thomassen [17]. The latter asks whether for fixed integers k and q the k -colorability of graphs of girth q on a fixed surface can be tested in polynomial time. (The other step concerns 4-coloring graphs on a fixed surface, and prospects for its resolution in the near future are not very bright at the moment.) With additional effort we will be able to implement our algorithm to run in linear time. Third, we will show that every triangle-free graph with an embedding of large edge-width in an orientable surface is 3-colorable. That generalizes a theorem of Hutchinson [10], who proved it under the stronger assumption that every face is even-sided.

More immediately, we apply Theorem 1.3 to prove another generalization of Grötzsch's theorem. Kawarabayashi and Thomassen [11] proved that there exists a function f such that every triangle-free graph G embedded in a surface of Euler genus g contains a set X of at most $f(g)$ vertices such that $G - X$ is 3-colorable. They prove the claim for a function $f(g) = O(g \log g)$, and believe that using a significantly more involved argument, they can improve the bound to linear (which is the best possible, as exemplified by a disjoint union of copies of the Grötzsch graph). Using the theory we develop, it is easy to prove this claim.

Theorem 1.4. *There exists a constant $\beta > 0$ with the following property. Every triangle-free graph G embedded in a surface of Euler genus g contains a set X of at most βg vertices such that $G - X$ is 3-colorable.*

The proof of Theorem 1.3 follows the method we developed in [4, 5]: we show that collapsing a 4-face (a standard reduction used when dealing with embedded triangle-free graphs) does not decrease a properly defined weight. By induction (with the base case given by the result of [5] for graphs of girth 5), this shows that the weight of every 4-critical embedded graph is bounded. As the contribution of each (≥ 5)-face to the weight is positive and linear in its size, the bound of Theorem 1.3 follows. Several technical difficulties are hidden in this brief exposition, however we worked out the solutions for them in [4, 5]. The main obstacle not encountered before is the need to avoid creating new non-facial 4-cycles in the reduction, which we deal with in Lemma 5.1.

Definitions and auxiliary results from [4, 5] that we are going to need in the proof are introduced in Sections 2 and 3. In Section 5, we prove a version of Theorem 1.3 generalized to allow some of the vertices to be precolored. Theorem 1.3 follows as a straightforward corollary as outlined in Section 4. Theorem 1.4 is proved in Section 6.

2 Definitions

All graphs in this paper are simple, with no loops or parallel edges.

A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to a circle, and we call it a *cuff*. For non-negative integers a , b and c , let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing the interiors of c pairwise disjoint closed discs. A standard result in topology shows that every surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a , b and c . Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The *Euler genus* $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff C of Σ , let \widehat{C} denote an open disk with boundary C such that \widehat{C} is disjoint from Σ , and let $\Sigma + \widehat{C}$ be the surface obtained by gluing Σ and \widehat{C} together, that is, by closing C with a patch. Let $\widehat{\Sigma} = \Sigma + \widehat{C}_1 + \dots + \widehat{C}_c$, where C_1, \dots, C_c are the cuffs of Σ , be the surface without boundary obtained by patching all the cuffs.

Consider a graph G embedded in the surface Σ ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G . We say that the embedding is *normal* if every cuff of Σ is equal to a cycle in G , and we call such a cycle a *ring*. Throughout the paper, all graphs are embedded normally. A *face* f of G is a maximal arcwise-connected subset of $\Sigma - G$. We write $F(G)$ for the set of faces of G . The boundary of a face is equal to a union of closed walks of G , which we call the *boundary walks* of f .

Consider a ring R . If R is a triangle and at most one vertex of R has degree greater than two in G , we say that R is a *vertex-like ring*. A ring with only vertices of degree two is *isolated*. For a vertex-like ring R that is not isolated, the *main* vertex of R is its vertex of degree greater than two. A vertex v of G is a *ring vertex* if v is contained in a ring (i.e., v is drawn in the boundary of Σ), and v is *internal* otherwise. A subgraph H of G is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ such that $H \subset \Delta$ (in particular, a cycle K is contractible if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K). The subgraph H *surrounds a cuff* C if H is not contractible in Σ , but it is contractible in $\Sigma + \widehat{C}$. We say that H *surrounds a ring* R if H surrounds the cuff incident with R . We say that H is *essential* if it is not contractible and it does not surround any cuff.

Let G be a graph embedded in a surface Σ , let the embedding be normal, and let \mathcal{R} be the set of rings of this embedding. In those circumstances we say that G is a *graph in Σ with rings \mathcal{R}* . Furthermore, some vertex-like rings are designated as *weak vertex-like rings*.

The *length* $|R|$ of a ring R is the number of vertices of R .¹ For a face f , by

¹Let us comment on a bit of a notational discrepancy between this paper and the previous papers of this series. In the previous papers, we have defined the length of vertex-like rings differently (the weak vertex-like rings had length 0, the non-weak ones length 1, while in this paper both are defined to have length 3). This was necessary for technical reasons that are not relevant in this paper's arguments, and we opt for the simpler definition here to avoid the

$|f|$ we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$). For a set of rings \mathcal{R} , let us define $\ell(\mathcal{R}) = \sum_{R \in \mathcal{R}} |R|$.

Let G be a graph with rings \mathcal{R} . Let $H = \bigcup \mathcal{R}$ and let H' be a (not necessarily induced) subgraph of G obtained from H by, for each weak vertex-like ring R , removing the main vertex and one of the non-main vertices of R (or by removing two vertices of R if R has no main vertex), so that H' intersects R in exactly one non-main vertex. A *precoloring* ψ of \mathcal{R} is a 3-coloring of the graph H' . A precoloring of \mathcal{R} *extends to a 3-coloring of G* if there exists a 3-coloring φ of G such that $\varphi(v) = \psi(v)$ for every $v \in V(H')$. The graph G is \mathcal{R} -critical if $G \neq H$ and for every proper subgraph G' of G that contains H , there exists a precoloring of \mathcal{R} that extends to a 3-coloring of G' , but not to a 3-coloring of G . For a precoloring κ of \mathcal{R} the graph G is κ -critical if κ does not extend to a 3-coloring of G , but it extends to a 3-coloring of every proper subgraph of G that contains \mathcal{R} .

Let us remark that if G is κ -critical for some κ , then it is \mathcal{R} -critical, but the converse is not true (for example, consider a graph consisting of a single ring with two chords). On the other hand, if κ is a precoloring of the rings of G that does not extend to a 3-coloring of G , then G contains a (not necessarily unique) κ -critical subgraph.

Let G be a graph in a surface Σ with rings \mathcal{R} . A face is *open 2-cell* if it is homeomorphic to an open disk. A face is *closed 2-cell* if it is open 2-cell and bounded by a cycle. A face f is *semi-closed 2-cell* if it is open 2-cell, and if a vertex v appears more than once in the boundary walk of f , then it appears exactly twice, v is the main vertex of a vertex-like ring R and the edges of R form part of the boundary walk of f . A face f is *omnipresent* if it is not open 2-cell and each of its boundary walks is a cycle bounding a closed disk $\Delta \subseteq \widehat{\Sigma} \setminus f$ containing exactly one ring.

3 Auxiliary results

First, let us state several simple properties of critical graphs (proofs can be found in [4]).

Lemma 3.1. *Let G be a graph in a surface Σ with rings \mathcal{R} . If G is \mathcal{R} -critical, then every internal vertex of G has degree at least three.*

Lemma 3.2. *Let G be a triangle-free graph in a surface Σ with rings \mathcal{R} . Suppose that each component of G is a planar graph containing exactly one of the rings. If G is \mathcal{R} -critical, then each component of G is 2-connected.*

Throughout the rest of the paper, let $s : \{2, 3, 4, \dots\} \rightarrow \mathbb{R}$ be the function defined by $s(l) = 0$ for $l \leq 4$, $s(5) = 4/4113$, $s(6) = 72/4113$, $s(7) = 540/4113$,

need to keep track of this special case. The only place where this change of definitions matters is in Theorem 3.4 taken from [5]. With the current definition of the vertex-like ring length, we claim a slightly weaker (and thus also correct) statement than the one proved in [5].

$s(8) = 2184/4113$ and $s(l) = l - 8$ for $l \geq 9$. Based on this function, we assign weights to the faces. Let G be a graph embedded in Σ with rings \mathcal{R} . For a face f of G , we define $w(f) = s(|f|)$ if f is open 2-cell and $w(f) = |f|$ otherwise. We define $w(G, \mathcal{R})$ as the sum of $w(f)$ over all faces f of G .

The main result of [4] bounds the weight of graphs embedded in the disk with one ring R (we write R -critical instead of $\{R\}$ -critical for brevity).

Theorem 3.3. *Let G be a graph of girth at least 5 embedded in the disk with one ring R of length $l \geq 5$. If G is R -critical, then $l \geq 8$ and $w(G, \{R\}) \leq s(l - 3) + s(5)$.*

This is extended to general surfaces in [5]. Let $\text{gen}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

$$\text{gen}(g, t, t_0, t_1) = 120g + 48t - 4t_1 - 5t_0 - 120.$$

Let $\text{surf}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1) + 116 - 42t = 8 - 4t_1 - 5t_0$ if $g = 0$ and $t = t_0 + t_1 = 2$,
- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1) + 114 - 42t = 6t - 4t_1 - 5t_0 - 6$ if $g = 0$, $t \leq 2$ and $t_0 + t_1 < 2$, and
- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1)$ otherwise.

A graph G embedded in a surface Σ with rings \mathcal{R} has *internal girth at least five* if every (≤ 4) -cycle in G is equal to one of the rings. Let $t_0(\mathcal{R})$ and $t_1(\mathcal{R})$ be the number of weak and non-weak vertex-like rings in \mathcal{R} , respectively.

Theorem 3.4 ([5, Theorem 6.2]). *There exists a constant η_0 with the following property. Let G be a graph embedded in a surface Σ with rings \mathcal{R} . If G is \mathcal{R} -critical and has internal girth at least five, then*

$$w(G, \mathcal{R}) \leq \eta_0 \cdot \text{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R}).$$

Consider a graph H embedded in a surface Π with rings \mathcal{Q} , and let f be a face of H . There exists a unique surface whose interior is homeomorphic to f , which we denote by Π_f . Note that the cuffs of Π_f correspond to the facial walks of f .

Let G be a graph embedded in Σ with rings \mathcal{R} . Let J be a subgraph of G and let S be a subset of faces of $J \cup \bigcup \mathcal{R}$ such that J is equal to the union of the boundaries of the faces in S . We define $G[S]$ to be the subgraph of G consisting of J and all the vertices and edges drawn inside the faces of S . Let C_1, C_2, \dots, C_k be the boundary walks of the faces in S . We would like to view $G[S]$ as a graph with rings C_1, \dots, C_k . However, the C_i 's do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty, we proceed as follows: Suppose that $S = \{f_1, \dots, f_m\}$.

For $1 \leq i \leq m$, let Σ_i be a surface with boundary B_i such that $\Sigma_i \setminus B_i$ is homeomorphic to f_i (i.e., Σ_i is homeomorphic to Σ_{f_i}). Let $\theta_i : \Sigma_i \setminus B_i \rightarrow f_i$ be a homeomorphism that extends to a continuous mapping $\theta_i : \Sigma_i \rightarrow \overline{f_i}$, where $\overline{f_i}$ denotes the closure of f_i . Let G_i be the inverse image of $G \cap \overline{f_i}$ under θ_i . Then G_i is a graph normally embedded in Σ_i . We say that the set of embedded graphs $\{G_i : 1 \leq i \leq m\}$ is a *G-expansion of S* (the subgraph J is uniquely determined by S as the union of the boundary walks of the faces in S). Note that there is a one-to-one correspondence between the boundary walks of the faces of S and the rings of the graphs in the G -expansion of S ; however, each vertex of J may be split to several copies. For $1 \leq i \leq m$, we let \mathcal{R}_i be the set of rings of G_i , where each vertex-like ring R is weak if and only if R is also a weak vertex-like ring of G . We say that the rings in \mathcal{R}_i are the *natural rings* of G_i .

We use the following basic property of critical graphs proved in [5].

Lemma 3.5. *Let G be a graph in a surface Σ with rings \mathcal{R} , and assume that G is \mathcal{R} -critical. Let J be a subgraph of G and S be a subset of faces of $J \cup \bigcup \mathcal{R}$ such that J is the union of the boundary walks of the faces of S . Let G' be an element of the G -expansion of S and let \mathcal{R}' be its natural rings. If G' is not equal to the union of the rings in \mathcal{R}' , then G' is \mathcal{R}' -critical.*

A frequently used corollary of Lemma 3.5 concerns the case that J is a contractible cycle.

Lemma 3.6. *Let G be a graph in a surface Σ with rings \mathcal{R} , and assume that G is \mathcal{R} -critical. Let C be a non-facial cycle in G bounding an open disk $\Delta \subseteq \Sigma$, and let G' be the graph consisting of the vertices and edges of G drawn in the closure of Δ . Then G' may be regarded as a graph embedded in the disk with one ring C , and as such it is C -critical.*

Together with the following following result of [7], this implies that every contractible (≤ 5)-cycle in a critical graph without contractible triangles bounds a face.

Theorem 3.7. *Let G be a triangle-free graph embedded in a disk with one ring R of length at most 6. If G is R -critical, then $|R| = 6$ and all faces of G have length 4. Furthermore, if $R = r_1 r_2 \dots r_6$ is an induced cycle and ψ is a precoloring of R that does not extend to a 3-coloring of G , then $\psi(r_1) = \psi(r_4)$, $\psi(r_2) = \psi(r_5)$, and $\psi(r_3) = \psi(r_6)$.*

Furthermore, criticality is also preserved when cutting the surface, which was again shown in [5].

Lemma 3.8. *Let G be a graph in a surface Σ with rings \mathcal{R} , and assume that G is \mathcal{R} -critical. Let c be a simple closed curve in Σ intersecting G in a set X of vertices. Let Σ_0 be one of the surfaces obtained from Σ by cutting along c . Let us split the vertices of G along c , let G' be the part of the resulting graph embedded in Σ_0 , let X' be the set of vertices of G' corresponding to the vertices*

of X and let $\mathcal{R}' \subseteq \mathcal{R}$ be the rings of G that are contained in Σ_0 . Let Δ be an open disk or a disjoint union of two open disks disjoint from Σ_0 such that the boundary of Δ is equal to the cuff(s) of Σ_0 corresponding to c . Let $\Sigma' = \Sigma_0 \cup \Delta$. Let Y consist of all vertices of X' that are not incident with a cuff in Σ' . For each $y \in Y$, choose an open disk $\Delta_y \subset \Delta$ such that the closures of the disks are pairwise disjoint and the boundary of Δ_y intersects G' exactly in y . Let $\Sigma'' = \Sigma' \setminus \bigcup_{y \in Y} \Delta_y$. For each $y \in Y$, add to G' a triangle R_y with $y \in V(R_y)$ tracing the boundary of Δ_y , and let $\mathcal{R}'' = \mathcal{R}' \cup \{R_y : y \in Y\}$, where the rings R_y are considered as non-weak vertex-like rings. If G' is not equal to the union of the rings in \mathcal{R}'' , then G' is \mathcal{R}'' -critical.

In particular, if G' is an component of a \mathcal{R} -critical graph, \mathcal{R}' are the rings contained in G' and G' is not equal to the union of \mathcal{R}' , then G' is \mathcal{R}' -critical.

Furthermore, the following claim was proved in [5] (in a stronger version allowing non-vertex-like rings of length 3).

Lemma 3.9. *Let G be an $\{R_1, R_2\}$ -critical graph embedded in the cylinder, where each of R_1 and R_2 is a vertex-like ring. If every cycle of length at most 4 in G is non-contractible, then G consists of R_1 , R_2 and an edge between them. In particular, neither R_1 nor R_2 is weak.*

We need to show that the same result holds even if we allow contractible 4-cycles.

Corollary 3.10. *Let G be an $\{R_1, R_2\}$ -critical graph embedded in the cylinder, where each of R_1 and R_2 is a vertex-like ring. If G contains no triangle distinct from the rings, then G consists of R_1 , R_2 and an edge between them. In particular, neither R_1 nor R_2 is weak.*

Proof. We proceed by induction, and thus we can assume that the claim holds for all graphs with less than $|V(G)|$ vertices. Without loss of generality, we can assume that neither R_1 nor R_2 is weak. Let φ be a precoloring of $\{R_1, R_2\}$ that does not extend to a 3-coloring of G .

Suppose first that the distance between R_1 and R_2 in G is at least four. By Lemmas 3.9 and 3.6 and by Theorem 3.7, we can assume that G contains a 4-face $f = v_1v_2v_3v_4$. For $i = 1, 2$, let G_i denote the graph obtained from G by identifying v_i with v_{i+2} and suppressing the arising parallel edges (observe that $v_iv_{i+2} \notin E(G)$, since G does not contain triangles distinct from the rings, and thus G_i does not contain loops). Since every 3-coloring of G_i extends to a 3-coloring of G , but φ does not extend to a 3-coloring of G , we conclude that φ does not extend to G_i . Since the distance between R_1 and R_2 in G is at least 4, G_i does not contain any edge between R_1 and R_2 , and by the induction hypothesis it follows that G_i contains a triangle.

Hence, for $i = 1, 2$, there exists a 5-cycle C_i in G containing the path $v_iv_{i+1}v_{i+2}$; that is, v_i is joined to v_{i+2} by a path of length exactly three in G . However, since G is embedded in the cylinder (a part of the plane), the existence of these paths implies that two adjacent vertices of f have degree at

least three and are contained in a triangle. This is a contradiction, since the only triangles in G are vertex-like rings.

Therefore, there exists a path P of length at most three between the main vertices r_1 and r_2 of R_1 and R_2 . Let P be the shortest such path. Let G_1 be the graph obtained from G by removing the non-main vertices of the rings, drawn in the sphere obtained by patching the cuffs incident with R_1 and R_2 . Let f be the face of the subgraph P of G_1 , and let G_2 be the unique element of the G_1 -expansion of $\{f\}$, with ring C of length at most 6. Since G is the shortest path between r_1 and r_2 , and since G_1 is triangle-free, the cycle C is induced.

Suppose for a contradiction that $G_2 \neq C$; similarly to Lemma 3.5, we can observe that then G_2 is C -critical. Since G_2 is triangle-free, Theorem 3.7 implies this is only possible if $|C| = 6$. Let $P = r_1 p_1 p_2 r_2$ and $C = r_1 p'_1 p'_2 r_2 p''_2 p'_1$. We can extend φ to a proper 3-coloring φ' of P ; we let ψ be a 3-coloring of C given by $\psi(r_1) = \varphi(r_1)$, $\psi(r_2) = \varphi(r_2)$, $\psi(p'_1) = \psi(p'_2) = \varphi'(p_1)$ and $\psi(p'_2) = \psi(p''_2) = \varphi'(p_2)$. By Theorem 3.7, ψ extends to a 3-coloring of G_2 , which implies that φ extends to a 3-coloring of G , which is a contradiction.

Therefore $G_2 = C$, and thus $G_1 = P$. Since G does not contain internal vertices of degree two by Lemma 3.1, it follows that G consists of R_1 , R_2 , and the edge $r_1 r_2$, as required. \square

4 The main result

For technical reasons, we are going to prove the following generalization of Theorem 3.4 instead of Theorem 1.3. We say that a graph G embedded in a surface Σ with rings \mathcal{R} is *internally triangle-free* if every triangle in G forms a vertex-like ring.

Theorem 4.1. *There exists a constant η with the following property. Let G be a graph embedded in a surface Σ with rings \mathcal{R} . If G is \mathcal{R} -critical, internally triangle-free, and contains no non-contractible 4-cycles, then*

$$w(G, \mathcal{R}) \leq \eta \cdot \text{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R}).$$

The following section is devoted to the proof of Theorem 4.1. Here, we show how it implies Theorem 1.3.

Proof of Theorem 1.3. Let η be the constant of Theorem 4.1; we can assume that $\eta \geq 1/80$. Let $\kappa = 1600\eta/s(5)$. Let $J \subseteq G$ be the union of all triangles and all 4-cycles that do not bound a 2-cell face. Note that $|E(J)| \leq 4c + 3t$. Let S be the set of all faces of J . Let $\{G_1, \dots, G_k\}$ be the G -expansion of S , where G_i is embedded in a surface Σ_i with rings \mathcal{R}_i for $1 \leq i \leq k$. Note that G_i is either equal to its rings or \mathcal{R}_i -critical, and that G_i has no vertex-like rings. Let the graph G'_i embedded in Σ_i with rings \mathcal{R}'_i be obtained from G_i by, for each ring $R \in \mathcal{R}_i$ of length at most 4, subdividing an edge of R by $5 - |R|$ new vertices. Observe that G'_i is either equal to its rings or \mathcal{R}'_i -critical, triangle-free and contains no non-contractible 4-cycles, and that $\ell(\mathcal{R}'_i) \leq 5\ell(\mathcal{R}_i)/3$. By

Theorem 4.1, we have

$$\begin{aligned}
z_i &:= \sum_{f \text{ face of } G'_i} (|f| - 4) \\
&\leq \frac{1}{s(5)} \sum_{f \text{ face of } G'_i} w(f) = \frac{1}{s(5)} w(G'_i, \mathcal{R}'_i) \\
&\leq \frac{120\eta(g(\Sigma_i) + |\mathcal{R}'_i| - 1) + \ell(\mathcal{R}'_i)}{s(5)} \\
&\leq \frac{120\eta(g(\Sigma_i) + \ell(\mathcal{R}'_i) - 1)}{s(5)},
\end{aligned}$$

because $\eta \geq 1/80$ and $\ell(\mathcal{R}'_i) \geq 3|\mathcal{R}'_i|$. Note that $\sum_{i=1}^k g(\Sigma_i) \leq g$ and $\sum_{i=1}^k \ell(\mathcal{R}'_i) \leq \frac{5}{3} \sum_{i=1}^k \ell(\mathcal{R}_i) = \frac{10|E(J)|}{3}$. We conclude that

$$\begin{aligned}
\sum_{f \text{ face of } G} (|f| - 4) &\leq \sum_{i=1}^k z_i \\
&\leq \frac{120\eta(g + 10|E(J)|/3 - 1)}{s(5)} \\
&\leq \frac{120\eta(g + 40c/3 + 10t - 1)}{s(5)} \\
&= \kappa \left((g + c + t - 1) + \frac{37}{40} - \frac{37}{40}g - \frac{1}{4}t \right).
\end{aligned}$$

Aksionov [1] proved that every planar graph with at most three triangles is 3-colorable; since G is 4-critical, we conclude that either $g \geq 1$ or $t \geq 4$, and in either case $\frac{37}{40} - \frac{37}{40}g - \frac{1}{4}t \leq 0$. Therefore,

$$\sum_{f \text{ face of } G} (|f| - 4) \leq \kappa(g + c + t - 1),$$

as desired. \square

5 The proof of Theorem 4.1

First, we show a key lemma that enables us to reduce 4-faces. Let us give the definitions we need to state this lemma.

Let Π be a surface with boundary and c a simple curve intersecting the boundary of Π exactly in its ends. The topological space obtained from Π by cutting along c (i.e., removing c and adding two new pieces of boundary corresponding to c) is a union of at most two surfaces. If surfaces Π_1, \dots, Π_k are obtained from Π by repeating this construction, we say that $\{\Pi_1, \dots, \Pi_k\}$ is a *fragmentation* of Π ; let us remark that zero repetitions of the construction are allowed, i.e., $\{\Pi\}$ is also a fragmentation of Π .

Let G and G' be graphs embedded in a surface Σ with rings \mathcal{R} , and let J be a subgraph of G containing $\bigcup \mathcal{R}$ as a proper subgraph. Suppose that there exists a collection $\{S_f : f \in F(G')\}$ of pairwise disjoint non-empty sets S_f of faces of J such that $F(J) = \bigcup_{f \in F(G')} S_f$ and for every face $f \in F(G')$, the set $\{\Sigma_h : h \in S_f\}$ is (up to homeomorphism) a fragmentation of Σ_f . In this situation, we say that J together with this collection $\{S_f : f \in F(G')\}$ forms a *cover of G by faces of G'* . We say that a set S_f is *non-trivial* if either $|S_f| > 1$, or $|S_f| = 1$ and the unique face of J in S_f is not a face of G (i.e., some vertices or edges of G are drawn within it). With respect to a fixed cover, we define the *elasticity* $\text{el}(f)$ of a face $f \in F(G')$ to be $\left(\sum_{h \in S_f} |h|\right) - |f|$.

Before proceeding further, let us roughly explain what we aim to do, thus motivating the above definitions; see Figure 1 for an illustration. We plan to prove Theorem 4.1 by induction on the complexity of the surface Σ and the size of G (formalized by the definition of an ordering \prec , which we introduce later). By Theorem 3.4, we can assume G contains a closed 2-cell 4-face h_0 bounded by a 4-cycle $v_1v_2v_3v_4$. Let G_0 be the graph obtained from G by identifying v_1 with v_3 to a new vertex z . Let us ignore for a moment a number of technical issues (triangles or non-contractible 4-cycles could be created by this identification, both v_1 and v_3 could be ring vertices) and suppose that G_0 is an internally triangle-free graph embedded in Σ with the same rings \mathcal{R} as G and without non-contractible 4-cycles. Let κ be a precoloring of \mathcal{R} that does not extend to a 3-coloring of G . Since every 3-coloring of G_0 extends to a 3-coloring of G , we conclude that κ does not extend to a 3-coloring of G_0 . Therefore, although the graph G_0 does not have to be \mathcal{R} -critical (e.g., in Figure 1, the vertex v_2 has degree two in G_0 , and thus it is irrelevant for 3-colorability), it must contain an \mathcal{R} -critical subgraph G' . By the induction hypothesis, we have $w(G', \mathcal{R}) \leq \eta \cdot \text{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R})$, and thus to prove Theorem 4.1, it suffices to argue that $w(G, \mathcal{R}) \leq w(G', \mathcal{R})$.

The graph G' is not quite a subgraph of G ; however, it can be transformed to a subgraph J of G by splitting the vertex z back into v_1 and v_3 and adding the path $v_1v_4v_3$. Each face f of G' then naturally corresponds to a set S_f of faces of J —usually, this set has only one element; however, it can happen that a face of G' is split during the construction of J , see e.g. the face f_1 in Figure 1, in which case $\{\Sigma_h : h \in S_f\}$ forms (up to homeomorphism) a fragmentation of Σ_f . It is easy to see that J together with the system $\{S_f : f \in F(G')\}$ forms a cover of G by faces of G' . Hence, to prove $w(G, \mathcal{R}) \leq w(G', \mathcal{R})$, we need to argue that for each $f \in F(G')$, the total weight of the faces of $G[S_f]$ is at most $w(f)$.

Let J_f denote the union of the boundaries of the faces in S_f . It is easy to see that since G is \mathcal{R} -critical, its subgraph $G[S_f]$ is critical with respect to J_f . There is a caveat: we may not be able to directly embed $G[S_f]$ in a surface (or surfaces, if $|S_f| > 1$) with rings corresponding to J_f ; e.g., suppose that Σ is a torus, and J_f is the union of two homotopic non-contractible cycles intersecting in one vertex w , and S_f consists of the face h of J_f homeomorphic to the open cylinder. Then the two boundary walks of h intersect, but distinct rings of a

graph must be vertex-disjoint. However, we can easily overcome this difficulty by splitting the vertex w into two vertices, so that J_f becomes a disjoint union of cycles and $G[S_f]$ can be naturally embedded in a cylinder with two rings (which is the surface we below denote by Σ_h). We formalized this splitting operation in Section 3 as the G -expansion of S_f , and argued it preserves criticality in Lemma 3.5.

Nevertheless, ignoring this subtlety, since $G[S_f]$ is J_f -critical, we can apply the induction hypothesis to it (ignoring yet another technical issue: to justify this step, we need to argue that the surfaces Σ_h for $h \in S_f$ are not more complex than Σ). If the sum of the lengths of the rings of $G[S_f]$ is the same as $|f|$, this directly gives the desired bound on the weights of faces in $G[S_f]$ in terms of $w(f)$. However, this does not need to be the case for the faces touching z ; in Figure 1, the face in S_{f_2} has length $|f_2| + 2$ (in terms defined above, f_2 has elasticity 2). A slightly more involved accounting is needed for such faces of non-zero elasticity.

Let us now proceed with a formal argument. As we indicated above, the described reduction is not valid for all choices of the 4-face h_0 . In particular, it may fail when h_0 contains ring vertices. Let G be a graph embedded in a surface Σ with rings \mathcal{R} . A 4-face $h_0 = v_1v_2v_3v_4$ is *ring-bound* if (up to a relabelling of the vertices of f)

- v_1 is the main vertex of a vertex-like ring R and there exists a cycle C of length at most 6 surrounding R such that $v_2v_3v_4 \subset C$; or,
- h_0 is incident with vertices of two distinct rings; or,
- v_1 and v_3 are ring vertices.

Another reason for the reduction to fail is that the identification of v_1 with v_3 creates a non-contractible 4-cycle. We cannot completely prevent this from happening, but we can at least ensure that such a 4-cycle has a quite special form. A non-contractible 4-cycle $C = w_1w_2w_3w_4$ is *flippable to a 4-face* if C surrounds a ring R , $w_1w_2w_3$ is part of a boundary walk of a face f_1 , $w_1w_4w_3$ is part of a boundary walk of a face f_2 , C separates f_1 from f_2 , and $w_1w_2w_3$ and $w_1w_4w_3$ are the only paths of length at most 2 in G between w_1 and w_3 . See the left part of Figure 2 for an illustration.

Lemma 5.1. *Let G be a graph embedded in a surface Σ with rings \mathcal{R} . Suppose that G is \mathcal{R} -critical and internally triangle-free, contains no non-contractible 4-cycles, and every connected essential subgraph of G has at least 13 edges. If G has a 4-face that is not ring-bound, then there exists an internally triangle-free \mathcal{R} -critical graph G' embedded in Σ with rings \mathcal{R} such that $|E(G')| < |E(G)|$, and a subgraph $J \subseteq G$ containing $\bigcup \mathcal{R}$ which together with a collection $\{S_f : f \in F(G')\}$ forms a cover of G by faces of G' , satisfying the following conditions.*

- *The graph G' has at most one non-contractible 4-cycle, and if it has one, it is flippable to a 4-face.*

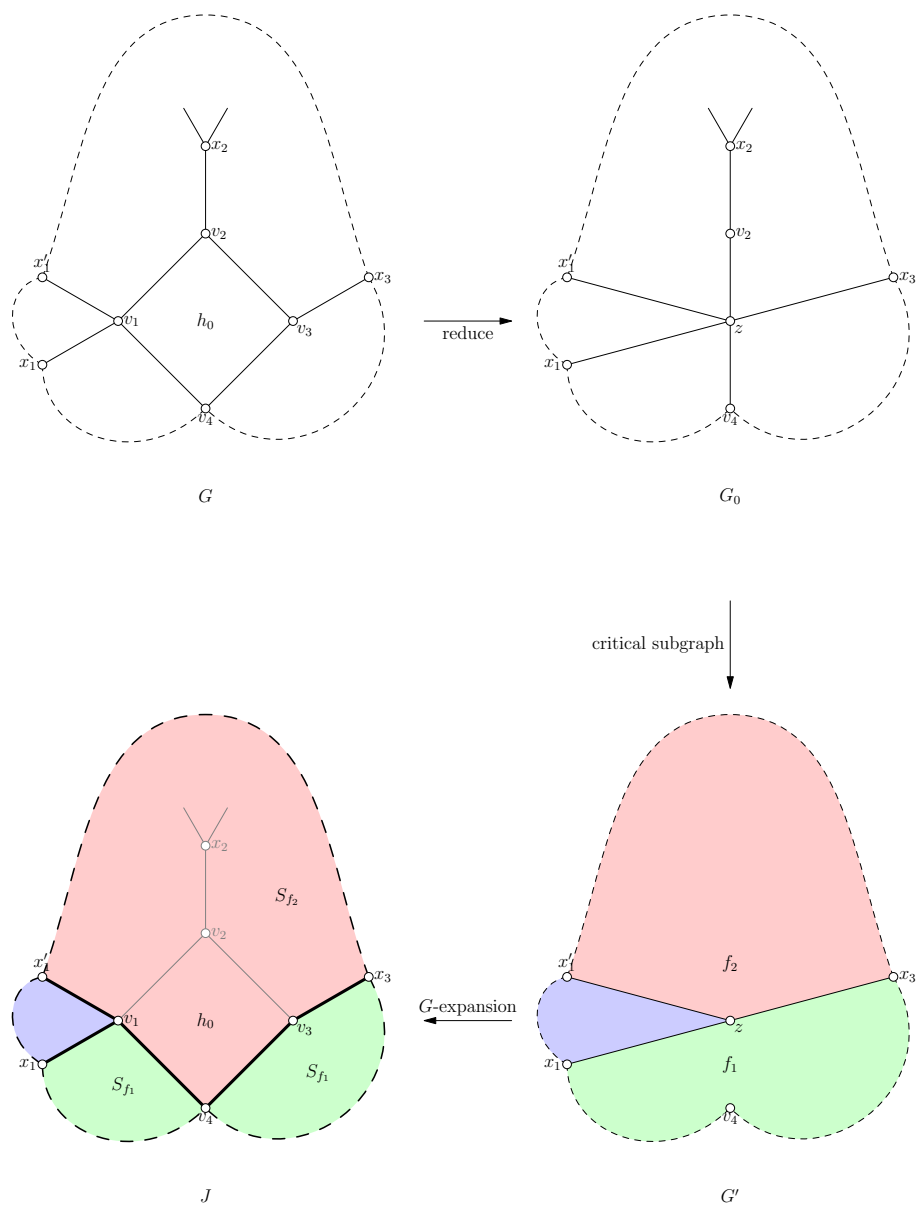


Figure 1: The main idea.

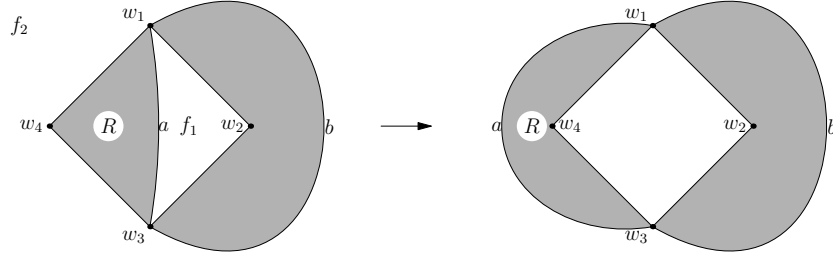


Figure 2: Non-contractible 4-cycle $w_1w_2w_3w_4$ flippable to a 4-face, and the corresponding transformation.

- The cover satisfies $\sum_{f \in F(G')} \text{el}(f) \leq 4$, and if a face $f \in F(G')$ is omnipresent or semi-closed 2-cell, then $\text{el}(f) \in \{0, 2\}$.
- For every face $f \in F(G')$, if f is semi-closed 2-cell and $\text{el}(f) = 2$, then S_f is non-trivial.

Proof. Let $h_0 = v_1v_2v_3v_4$ be a 4-face of G that is not ring-bound. In particular, v_1 and v_3 are not both ring vertices, and v_2 and v_4 are not both ring vertices; by symmetry, we can assume that v_3 and v_4 are internal vertices. By Lemma 3.1, there exist edges e_3 and e_4 incident with v_3 and v_4 , respectively, and not incident with h_0 .

As G is internally triangle-free, v_1 and v_3 are non-adjacent and v_2 and v_4 are non-adjacent. For $i \in \{1, 2\}$, we say that the vertex v_i is *problematic* if either v_i is the main vertex of a vertex-like ring, or G contains a path P_i of length at most four joining v_i with v_{i+2} such that the concatenation of the paths P_i and $v_iv_{5-i}v_{i+2}$ is a non-contractible cycle C_i . By switching the labels of vertices if necessary, we can assume that if v_1 is problematic, then v_2 is problematic as well.

Suppose that v_1 is the main vertex of a vertex-like ring R . Then v_1 is problematic, and thus v_2 is problematic. As h_0 is not ring-bound, v_2 is an internal vertex, and thus G contains a non-contractible cycle C_2 described above. Since h_0 is not ring-bound, we conclude C_2 does not surround v_1 . However, then C_2 together with R and the edge v_1v_2 forms a connected essential subgraph of G with at most 11 edges, which is a contradiction. We conclude that v_1 is not the main vertex of a vertex-like ring; and by symmetry, if both v_1 and v_2 are problematic, then v_2 is not the main vertex of a vertex-like ring.

Let G_0 be a multigraph obtained from G by deleting edges v_2v_3 and v_3v_4 and identifying v_1 with v_3 to a new vertex z (parallel edges may arise if v_1 and v_3 have common neighbors other than v_2 and v_4). Since the vertex v_3 is internal, G_0 is embedded in Σ with rings \mathcal{R} . Furthermore, note that each edge $e \in E(G_0)$ corresponds to a unique edge \bar{e} of G . Every 3-coloring of G_0 corresponds to a 3-coloring of G in which v_1 and v_3 have the same color as z . Let ψ be a precoloring of \mathcal{R} that does not extend to G , and observe that ψ also does not extend to G_0 .

Therefore, G_0 has a ψ -critical subgraph G' (the criticality implies G' does not have parallel edges).

If there exists a path xzy in G_0 contained in the boundary of a face such that \overline{xz} is incident with v_1 in G and \overline{yz} is incident with v_3 in G , then let $A = \{v_1v_4, v_3v_4\}$, otherwise let $A = \emptyset$. Let J be the subgraph of G with edge set $\{\overline{e} : e \in E(G')\} \cup A$ and with the vertex set consisting of all vertices incident with these edges (informally, we split the vertex z back into v_1 and v_3 , and if that breaks the boundary of any face, we fix it by adding the path $v_1v_4v_3$). Each face f of G' naturally corresponds to either one or two faces of J (the latter happens when $v_4 \in V(G')$, z is not adjacent to v_4 in G' , and $A \neq \emptyset$, see the face f_1 in Figure 1 for an illustration); let S_f denote the set of faces of J corresponding to f . The construction implies that $\{\Sigma_h : h \in S_f\}$ is (up to homeomorphism) a fragmentation of Σ_f . Furthermore, every face of J is contained in S_f for a unique face $f \in F(G')$. Therefore, J together with the collection $\{S_f : f \in F(G')\}$ forms a cover of G by faces of G' .

Observe that a face $f \in F(G')$ has non-zero elasticity if and only if a facial walk of f contains two consecutive edges e_1 and e_2 incident with z such that $\overline{e_1}$ is incident with v_1 in G , $\overline{e_2}$ is incident with v_3 in G , and $\overline{e_1} \neq v_1v_4$; and each such pair of edges contributes 2 to the elasticity of f . There are at most two such pairs of edges in the cyclic order around z , and thus the sum of the elasticities of faces of G' is at most 4. Furthermore, if both of these pairs of edges are incident with the same face f , then z is incident twice with f ; since z is not the main vertex of a vertex-like ring of G' (as neither v_1 nor v_3 is the main vertex of a vertex-like ring in G , and G has no non-vertex-like rings of length 3), it follows that the face f of G' is neither semi-closed 2-cell nor omnipresent. We conclude that if a face $f \in F(G')$ is semi-closed 2-cell or omnipresent, then its elasticity is at most two.

Suppose that f is a semi-closed 2-cell face of G' , the elasticity of f is not 0, and $|S_f| = 1$ (let h be the unique face in S_f). Then the boundary of h contains the path $v_1v_4v_3$. Note that $v_2v_3 \notin E(J)$, and thus if h_0 is a subset of h , then an edge of G is drawn within h . If h_0 is not a subset of h , then the edge e_4 of G is drawn within h . We conclude that for every semi-closed 2-cell face f of G' with $\text{el}(f) > 0$, the set S_f is non-trivial.

To finish the proof of Lemma 5.1, it suffices to argue that G' is internally triangle-free, has at most one non-contractible 4-cycle, and if it has one, it is flippable to a 4-face. Hence, suppose that G' contains either a triangle which is not a vertex-like ring, or a non-contractible 4-cycle. Then G contains a path P_1 of length 3 or 4 joining v_1 with v_3 such that P_1 together with $v_1v_4v_3$ forms a cycle C_1 such that if $|C_1| = 6$, then C_1 is non-contractible. Let us first consider the case that $|C_1| = 5$ and C_1 is contractible, and let $\Delta \subset \Sigma$ be a closed disk bounded by C_1 . Since $v_1v_4v_3$ is a subpath of C_1 , either the edge e_4 or the 4-face h_0 of G is contained in the interior of Δ . Consequently, Lemma 3.6 implies that the subgraph of G drawn in Δ is C_1 -critical. However, as $|C_1| = 5$, this contradicts Theorem 3.7.

We conclude that C_1 is non-contractible, and thus v_1 is problematic. By the choice of the labels of vertices incident with h_0 , it follows that v_2 is problematic

as well. We argued before that if both v_1 and v_2 are problematic, then v_2 is not the main vertex of a vertex-like ring; and thus there exists a path P_2 joining v_2 with v_4 which combines with the path $v_2v_3v_4$ to a non-contractible (≤ 6)-cycle C_2 .

Note that the connected subgraph H of G formed by the cycle $v_1v_2v_3v_4$ and the paths P_1 and P_2 has at most 12 edges, and thus it is not essential. It follows that there exists a cuff C (incident with a ring R) and a closed disk $\Lambda \subseteq \Sigma + \widehat{C}$ with H contained in Λ . Observe that H does not contain a vertex-like ring, and since G is internally triangle-free, it follows that H is triangle-free. Extend the disk Λ to a plane and consider the resulting drawing of H in the plane; for $1 \leq i \leq 4$, let h_i be the face of H in this plane drawing incident with $v_i v_{i+1}$ (where $v_5 = v_1$) and distinct from h_0 . The faces h_1, \dots, h_4 are pairwise distinct, as any two of them are separated by C_1 or C_2 . Since H is triangle-free, all faces of H have length at least four. Furthermore, observe that the boundary walks of two faces of H (the face containing \widehat{C} and the unbounded face) surround R , and since G does not contain non-contractible 4-cycles, it follows that these two faces have length at least 5. Therefore,

$$22 \leq \sum_{h \in F(H)} |h| = 8 + 2|E(P_1) \cup E(P_2)| \leq 24. \quad (1)$$

In particular, H has no face distinct from h_0, \dots, h_4 . By symmetry, we can assume that \widehat{C} is contained in h_3 , and thus $|h_3| \geq 5$. The cycles C_1 and C_2 are non-contractible, and thus they surround R ; since h_1 is the only face of H distinct from h_0 separated from h_3 by both C_1 and C_2 , it follows that h_1 is the unbounded face of H . Hence, the boundary walk of h_1 surrounds R and $|h_1| \geq 5$. Note that $|h_1| + |h_3| \leq |E(P_1)| + |E(P_2)| + 2 \leq 10$, and thus $|h_1| = |h_3| = 5$ and $|E(P_1)| = |E(P_2)| = 4$. As the subdisk of Λ bounded by C_2 consists of the closure of the faces h_2 and h_3 and $|C_2| = 6$, it follows that $|h_2|$ and $|h_3|$ have the same parity, i.e., $|h_2|$ is odd. Similarly, $|h_4|$ is odd. We conclude that $|h_2|, |h_4| \geq 5$, and by (1), it follows that $|h_2| = |h_4| = 5$ and the paths P_1 and P_2 are edge-disjoint. That is, the P_1 and P_2 intersect in a single vertex w_1 . Observe that since $|h_2| = |h_3| = |h_4|$, w_1 is the middle vertex of P_1 and P_2 .

Since C_1 and C_2 surround R , the cycles bounding h_2 and h_4 are contractible, and thus by Lemma 3.6 and Theorem 3.7, h_2 and h_4 are faces of G . Let $Q_1 = v_1 w_2 w_1$ be the subpath of P_1 between v_1 and w_1 . If G contained another path Q'_1 of length at most two between v_1 and w_1 , then the cycle consisting of Q_1 and Q'_1 would be a non-ring triangle, or a non-contractible 4-cycle, or a 4-face showing that w_2 is an internal vertex of degree two, which is a contradiction. Therefore, Q_1 is the only path of length at most two between v_1 and w_1 in G . Similarly, the subpath $v_3 w_4 w_1$ of P_2 is the only path of length at most two between v_3 and w_1 in G .

We conclude that the graph G' is internally triangle-free and contains (at most) one non-contractible 4-cycle $w_1 w_2 w_3 w_4$, where $w_3 = z$. Because of the faces h_2 and h_4 of G , the paths $w_1 w_2 w_3$ and $w_1 w_4 w_3$ are parts of boundaries

of faces of G' separated by $w_1w_2w_3w_4$. Furthermore, $w_1w_2w_3w_4$ surrounds the ring R . We conclude that $w_1w_2w_3w_4$ is flippable to a 4-face. \square

As the next step, let us prove a stronger variant of Theorem 4.1 in the case Σ is the disk. Let $\mathcal{G}_{r,k}$ denote the class of all R -critical graphs of girth at least r embedded in the disk with one ring R of length k . For a graph $G \in \mathcal{G}_{4,k}$ let $S(G)$ denote the multiset of lengths of the (≥ 5) -faces of G . Let $\mathcal{S}_{r,k} = \{S(G) : G \in \mathcal{G}_{r,k}\}$. Note that by Theorem 3.3, the set $\mathcal{S}_{5,k}$ is empty for $5 \leq k \leq 7$ and finite for every $k \geq 8$, the maximum of each element of $\mathcal{S}_{5,k}$ is at most $k-3$, and if the maximum of $S \in \mathcal{S}_{5,k}$ is equal to $k-3$, then $S = \{5, k-3\}$. Furthermore, $\mathcal{S}_{4,4} = \mathcal{S}_{4,5} = \emptyset$ and $\mathcal{S}_{4,6} = \{\emptyset\}$ by Theorem 3.7.

Let S_1 and S_2 be multisets of integers. We say that S_2 is a *one-step refinement* of S_1 if there exists $k \in S_1$ and a multiset $Z \in \mathcal{S}_{4,k} \cup \mathcal{S}_{4,k+2}$ such that $S_2 = (S_1 \setminus \{k\}) \cup Z$. We say that S_2 is a *refinement* of S_1 if it can be obtained from S_1 by a (possibly empty) sequence of one-step refinements.

Lemma 5.2. *For every $k \geq 7$, each element of $\mathcal{S}_{4,k}$ other than $\{k-2\}$ is a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. In particular, if $S \in \mathcal{S}_{4,k}$, then the maximum of S is at most $k-2$, and if the maximum is equal to $k-2$, then $S = \{k-2\}$.*

Proof. Let us re-phrase the claim.

For every graph $G \in \mathcal{G}_{4,k}$, either $S(G) = \{k-2\}$ or $S(G)$ is a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. (2)

We prove this reformulation by induction on the number of edges of G ; that is, we assume that (2) holds for every graph $G' \in \mathcal{G}_{4,k}$ with $|E(G')| < |E(G)|$.

Let us remark that this implies that the maximum of each element of $\mathcal{S}_{4,k}$ distinct from $\{k-2\}$ is at most $k-3$, by induction on k . Indeed, assuming this claim holds for every $k' < k$, the maximum of each element $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$ is at most $k-3$ by Theorem 3.3, and thus it suffices to argue that this property is preserved by one-step refinements. This is the case, since in the one-step refinement operation, we are replacing an element $c \leq k-3$ by elements of $\mathcal{S}_{4,c} \cup \mathcal{S}_{4,c+2}$, which are all smaller or equal to $(c+2)-2 \leq k-3$ by the assumption.

Let us now proceed with the proof of (2). By Theorem 3.7, (2) holds if $k \leq 6$, and thus we can assume that $k \geq 7$. Consider a graph $G \in \mathcal{G}_{4,k}$. Note that every 4-cycle in G bounds a face by Lemma 3.6 and Theorem 3.7. If G has no 4-faces, then $S(G) \in \mathcal{S}_{5,k}$ and (2) holds. Hence, suppose G contains a 4-face $h_0 = v_1v_2v_3v_4$.

Let us first consider the case that h_0 is ring-bound. Since G has only one ring and this ring is not vertex-like, we can assume both v_1 and v_3 belong to the ring R of G . Consequently G contains a path P of length 1 or 2 intersecting R exactly in the endpoints of P . Let C_1 and C_2 be the cycles of $R+P$ distinct from R , and let G_1 and G_2 be the subgraphs of G drawn in the closed disks bounded by C_1 and C_2 , respectively. We have $|C_1| + |C_2| = k + 2|E(P)|$. Suppose that

say $|C_1| = 4$; hence, C_1 bounds a face by Lemma 3.6 and Theorem 3.7, and $S(G) = S(G_2)$. If C_2 bounds a face, then $|E(P)| = 1$, as otherwise the middle vertex of P would contradict Lemma 3.1, and $S(G) = \{k-2\}$, implying (2). Hence, we can assume C_2 does not bound a face, and thus G_2 is C_2 -critical by Lemma 3.6. If $|E(P)| = 2$, then $|C_2| = k$, and since $S(G) = S(G_2)$ and $|E(G_2)| < |E(G)|$, (2) follows by the induction hypothesis for G_2 . If $|E(P)| = 1$, then $|C_2| = k-2$ and $S(G) = S(G_2) \in \mathcal{S}_{4,k-2}$, and thus (2) holds.

Therefore, we can assume that $|C_1| \geq 5$ and by symmetry, $|C_2| \geq 5$. If $|E(P)| = 1$, then $|C_1| + |C_2| = k+2$, and thus $Z = \{|C_1|, |C_2|\}$ belongs to $\mathcal{S}_{5,k}$ (the element corresponds to a ring of length k with a chord). Since $S(G_i)$ is either $\{|C_i|\}$ or belongs to $\mathcal{S}_{4,|C_i|}$ for $i \in \{1, 2\}$, the multiset $S(G) = S(G_1) \cup S(G_2)$ is a refinement of the element Z of $\mathcal{S}_{5,k}$, implying (2). Finally, consider the case that $|E(P)| = 2$. As the middle vertex of P has degree at least three, by symmetry we can assume that C_2 does not bound a face, and thus G_2 is C_2 -critical. In particular $|C_2| \geq 6$. If $|C_2| = 6$, then $S(G_2) = \emptyset$ by Theorem 3.7 and $|C_1| = k-2$. Consequently $S(G) = S(G_1)$ and either $S(G_1) = \{k-2\}$ or $S(G_1)$ belongs to $\mathcal{S}_{4,k-2}$, implying (2). If $|C_2| \geq 7$, then note that $|C_1| + (|C_2| - 2) = k+2$, and thus $Z' = \{|C_1|, |C_2| - 2\}$ belongs to $\mathcal{S}_{5,k}$. Furthermore, $S(G_2) \in \mathcal{S}_{4,|C_2|} = \mathcal{S}_{4,|C_2|-2+2}$, and thus $\{C_1\} \cup S(G_2)$ is a one-step refinement of Z' . Consequently, $S(G)$ is a refinement of the element Z' of $\mathcal{S}_{5,k}$, implying (2).

Therefore, we can assume h_0 is not ring-bound. As G is embedded in the disk, G has no essential subgraph, and thus Lemma 5.1 applies; let G' be the corresponding R -critical graph, and let $J \subseteq G$ together with a system $\{S_f : f \in F(G')\}$ be the cover of G by faces of G' . By Lemma 3.2, each face of G' is closed 2-cell, and thus each face of G' has elasticity at most 2.

For each face $f \in F(G')$, the G -expansion of S_f consists of $t = |S_f|$ graphs H_1, \dots, H_t embedded in disks with rings R_1, \dots, R_t such that $|R_1| + \dots + |R_t| = |f| + \text{el}(f)$ and $|R_1| \geq \dots \geq |R_t|$. Let $m \in \{0, \dots, t\}$ be the index such that $|R_i| > 4$ for $1 \leq i \leq m$ and $|R_i| = 4$ for $m+1 \leq i \leq t$. Observe that if $t \geq 1 + \text{el}(f)/2$, then $\{|R_1|, \dots, |R_m|\}$ is an element of $\mathcal{S}_{4,|f|}$ (corresponding to the graph obtained from a ring of length $|f|$ by adding $t-1$ chords to form faces of lengths $|R_1|, \dots, |R_{t-1}|$, and then adding further chords to split the last face of length $|f| + 2(t-1) - |R_1| - \dots - |R_{t-1}| = |R_t| + 2(t-1) - \text{el}(f)$ into a face of length $|R_t|$ and 4-faces), and thus $S(H_1) \cup \dots \cup S(H_t)$ is a refinement of $\{|f|\}$. If $t < 1 + \text{el}(f)/2$, then $t = 1$ and $\text{el}(f) = 2$ and Lemma 5.1 asserts that S_f is non-trivial. In that case, it follows that H_1 is R_1 -critical and $|R_1| = |f| + 2$, and thus $S(H_1)$ belongs to $\mathcal{S}_{4,|f|+2}$; and again, $S(H_1)$ is a refinement of $\{|f|\}$. Since this holds for every $f \in F(G')$ and J with $\{S_f : f \in F(G')\}$ forms a cover of G by faces of G' , we conclude that $S(G)$ is a refinement of $S(G')$.

Since $|E(G')| < |E(G)|$, the induction hypothesis implies that either $S(G') = \{k-2\}$ or $S(G')$ is a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. In the latter case, since $S(G)$ is a refinement of $S(G')$, it follows that $S(G)$ is also a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$, and (2) holds. Hence, suppose that $S(G') = \{k-2\}$. Let f be the $(k-2)$ -face of G' and let H_1, \dots, H_t and m be as in the previous paragraph; we have $S(G) = S(H_1) \cup \dots \cup S(H_m)$. By the argument from the

previous paragraph, if $t \geq 1 + \text{el}(f)/2$, this implies $S(G)$ is a refinement of an element $\{|R_1|, \dots, |R_m|\}$ of $\mathcal{S}_{4,|f|} = \mathcal{S}_{4,k-2}$. Hence, suppose that $t = 1$ and $\text{el}(f) = 2$, and thus S_f is non-trivial. Hence, H_1 is R_1 -critical and $|R_1| = k$. Note that $|E(H_1)| < |E(G)|$, since R is a proper subgraph of J by the definition of a cover by faces. Since $S(G) = S(H_1)$, (2) follows from the induction hypothesis applied to H_1 . \square

For a multiset Z , let $s(Z) = \sum_{z \in Z} s(z)$.

Corollary 5.3. *Let G be a triangle-free graph embedded in the disk with one ring R of length $l \geq 4$. If G is R -critical, then $w(G, \{R\}) \leq s(l-2)$.*

Proof. We proceed by induction on l . The claim holds for $l \leq 6$ by Theorem 3.7. Suppose that $l \geq 7$. By Lemma 5.2, we have that $S(G)$ is either $\{l-2\}$ or a refinement of $Z \in \mathcal{S}_{4,l-2} \cup \mathcal{S}_{5,l}$. In the former case, we have $w(G, \{R\}) = s(l-2)$, and thus we can assume the latter.

If $Z \in \mathcal{S}_{4,l-2}$, then $s(Z) \leq s(l-4) < s(l-2)$ by the induction hypothesis. If $Z \in \mathcal{S}_{5,l}$, then $l \geq 8$ and $s(Z) \leq s(l-3) + s(5) < s(l-2)$ by Theorem 3.3. Since $S(G)$ is a refinement of Z , there exist multisets $Z = Z_0, Z_1, \dots, Z_m = S(G)$ such that Z_{i+1} is a one-step refinement of Z_i for $0 \leq i \leq m-1$. We claim that $s(Z_i) < s(l-2)$ for $i = 1, \dots, m$. Indeed, assuming that the claim holds for $i-1$, the maximum of Z_{i-1} is at most $l-3$, and since Z_i is a one-step refinement of Z_{i-1} , we have $s(Z_i) = s(Z_{i-1}) - s(a) + s(A)$ for some $a \in Z_{i-1}$ and $A \in \mathcal{S}_{4,a} \cup \mathcal{S}_{4,a+2}$. Since $a+2 < l$, by the induction hypothesis we have $s(A) \leq s((a+2)-2) = s(a)$, and thus $s(Z_i) \leq s(Z_{i-1}) < s(l-2)$. We conclude that $S(G) = s(Z_m) < s(l-2)$. \square

We will need the following properties of the function surf , which were proved in [5].

Lemma 5.4. *If $g, g', t, t_0, t_1, t'_0, t'_1$ are non-negative integers, then the following holds:*

- (a) *Assume that if $g = 0$ and $t \leq 2$, then $t_0 + t_1 < t$. If $t \geq 2$, $t'_0 \leq t_0$, $t'_1 \leq t_1$ and $t'_0 + t'_1 \geq t_0 + t_1 - 2$, then $\text{surf}(g, t-1, t'_0, t'_1) \leq \text{surf}(g, t, t_0, t_1) - 1$.*
- (b) *If $g' < g$ and either $g' > 0$ or $t \geq 2$, then $\text{surf}(g', t, t_0, t_1) \leq \text{surf}(g, t, t_0, t_1) - 120(g - g') + 32$.*
- (c) *Let g'', t', t'', t'_0 and t'_1 be nonnegative integers satisfying $g = g' + g''$, $t = t' + t''$, $t_0 = t'_0 + t''_0$, $t_1 = t'_1 + t''_1$, either $g'' > 0$ or $t'' \geq 1$, and either $g' > 0$ or $t' \geq 2$. Then, $\text{surf}(g', t', t'_0, t'_1) + \text{surf}(g'', t'', t''_0, t''_1) \leq \text{surf}(g, t, t_0, t_1) - \delta$, where $\delta = 16$ if $g'' = 0$ and $t'' = 1$ and $\delta = 56$ otherwise.*
- (d) *If $g \geq 2$, then $\text{surf}(g-2, t, t_0, t_1) \leq \text{surf}(g, t, t_0, t_1) - 124$.*

Consider a graph H embedded in a surface Π with rings \mathcal{Q} , and let f be a face of H . Let a_0 and a_1 be the number of weak and non-weak vertex-like rings, respectively, that form one of the facial walks of f by themselves. Let a be the number of facial walks of f . We define $\text{surf}(f) = \text{surf}(g(\Pi_f), a, a_0, a_1)$.

Let G_1 be a graph embedded in Σ_1 with rings \mathcal{R}_1 and G_2 a graph embedded in Σ_2 with rings \mathcal{R}_2 . Let $m(G_i)$ denote the number of edges of G_i that are not contained in the boundary of Σ_i . Let us write $(G_1, \Sigma_1, \mathcal{R}_1) \prec (G_2, \Sigma_2, \mathcal{R}_2)$ to denote that the quadruple $(g(\Sigma_1), |\mathcal{R}_1|, m(G_1), |E(G_1)|)$ is lexicographically smaller than $(g(\Sigma_2), |\mathcal{R}_2|, m(G_2), |E(G_2)|)$.

We now have everything needed to prove Theorem 4.1.

Proof of Theorem 4.1. Let η_0 be the constant from Theorem 3.4 and let $\eta = 6\eta_0 + 44$. We proceed by induction and assume that the claim holds for all graphs G' embedded in surfaces Σ' with rings \mathcal{R}' such that $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$. Let $g = g(\Sigma)$, $t_0 = t_0(\mathcal{R})$ and $t_1 = t_1(\mathcal{R})$. By Theorem 1.1, Corollary 5.3 and Theorem 1.2, the claim holds if $g + |\mathcal{R}| \leq 1$, hence assume that $g + |\mathcal{R}| \geq 2$. Similarly, if $g = 0$ and $|\mathcal{R}| = 2$, then we can assume that $t_0 + t_1 \leq 1$ by Corollary 3.10. First, we aim to prove that we can assume that all faces in the embedding of G are semi-closed 2-cell. For later use, we will consider a more general setting.

Let H be an internally triangle-free graph embedded in a surface Π with rings \mathcal{Q} without non-contractible 4-cycles, such that at least one face of H is not open 2-cell and no face of H is omnipresent. If H is \mathcal{Q} -critical and $(H, \Pi, \mathcal{Q}) \preceq (G, \Sigma, \mathcal{R})$, then

$$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + \eta \left(\text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 7 - \sum_{h \in F(H)} \text{surf}(h) \right). \quad (3)$$

Proof. We prove the claim by induction. Consider for a moment an internally triangle-free graph H' embedded in a surface Π' with rings \mathcal{Q}' without non-contractible 4-cycles, where $(H', \Pi', \mathcal{Q}') \prec (H, \Pi, \mathcal{Q})$, such that either $H' = \mathcal{Q}'$ or H' is \mathcal{Q}' -critical. We claim that

$$w(H', \mathcal{Q}') \leq \ell(\mathcal{Q}') + \eta \left(\text{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \text{surf}(h) \right). \quad (4)$$

The claim obviously holds if $H' = \mathcal{Q}'$, since $w(H', \mathcal{Q}') \leq \ell(\mathcal{Q}')$ in that case; hence, it suffices to consider the case that H' is \mathcal{Q}' -critical. If at least one face of H' is not open 2-cell and no face of H' is omnipresent, then this follows by an inductive application of (3) (we could even strengthen the inequality by 7η). If all faces of H' are open 2-cell, then note that $\text{surf}(h) = 0$ for every $h \in H'$, and since $(H', \Pi', \mathcal{Q}') \prec (G, \Sigma, \mathcal{R})$, we can apply Theorem 4.1 inductively to obtain (4). Finally, suppose that H' has an omnipresent face f , let $\mathcal{Q}' = \{Q_1, \dots, Q_t\}$ and for $1 \leq i \leq t$, let \widehat{C}_i be the cuff traced by Q_i , let Δ_i be a closed disk in $\Pi' + \widehat{C}_i$ such that $\widehat{C}_i \subset \Delta_i$ and the boundary of Δ_i is a subset of f , and let f_i denote the boundary walk of f contained in Δ_i . Since all components of H' are planar and contain only one ring, Lemma 3.2 implies that all faces

of H' distinct from f are closed 2-cell. Furthermore, by Theorem 3.7, each vertex-like ring forms component of the boundary of f by itself, and hence $\text{surf}(f) = \text{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}'))$. If Q_i is not a vertex-like ring, then by applying Corollary 5.3 to the subgraph H'_i of H' embedded in $\widehat{\Delta}_i \setminus \widehat{C}_i$, we conclude that the weight of H'_i is at most $s(|Q_i|)$ and that $|f_i| \leq |Q_i|$. Note that $s(|Q_i|) - s(|f_i|) \leq |Q_i| - |f_i|$. Therefore, we again obtain (4):

$$\begin{aligned} w(H', \mathcal{Q}') &\leq |f| + \sum_{i=1}^t s(|Q_i|) - s(|f_i|) \leq |f| + \sum_{i=1}^t |Q_i| - |f_i| \\ &= \ell(\mathcal{Q}') \\ &= \ell(\mathcal{Q}') + \eta \left(\text{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \text{surf}(h) \right). \end{aligned}$$

Let us now return to the graph H . Since H is \mathcal{Q} -critical, Theorem 1.1 implies that no component of H is a planar graph without rings. Let f be a face of H which is not open 2-cell. By the assumptions, f is not omnipresent, and thus $g(\Pi) > 0$ or $|\mathcal{Q}| > 2$. Let c be a simple closed curve in f infinitesimally close to a facial walk W of f . Cut Π along c and cap the resulting holes by disks (c is always a 2-sided curve). Let Π_1 be the surface obtained this way that contains W , and if c is separating, then let Π_2 be the other surface. Since f is not omnipresent, we can choose W so that either $g(\Pi_1) > 0$ or Π_1 contains at least two rings of \mathcal{Q} . Let us discuss several cases:

- *c is separating and H is contained in Π_1 .* Let H_1 denote the embedding of H in Π_1 . In this case f has only one facial walk, and since f is not open 2-cell, Π_2 is not the sphere. It follows that $g(\Pi_1) = g(\Pi) - g(\Pi_2) < g(\Pi)$, and thus $(H_1, \Pi_1, \mathcal{Q}) \prec (H, \Pi, \mathcal{Q})$. Note that the weights of the faces of the embedding of H in Π and of H_1 in Π_1 are the same, with the exception of f whose weight in Π is $|f|$ and the weight of the corresponding face of H_1 in Π_1 is $s(|f|) \geq |f| - 8$. By (4), we have

$$\begin{aligned} w(H, \mathcal{Q}) &\leq w(H_1, \mathcal{Q}) + 8 \\ &\leq \ell(\mathcal{Q}) + 8 + \eta \left(\text{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \text{surf}(f) - \sum_{h \in F(H)} \text{surf}(h) \right). \end{aligned}$$

Note that $\text{surf}(f) = 120g(\Pi_2) - 72$. By Lemma 5.4(b), we conclude that

$$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 8 + \eta \left(\text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 40 - \sum_{h \in F(H)} \text{surf}(h) \right).$$

- *c is separating and Π_2 contains a nonempty part H_2 of H .* Let H_1 be the part of H contained in Π_1 . Let \mathcal{Q}_i be the subset of \mathcal{Q} belonging to Π_i and f_i the face of H_i corresponding to f , for $i \in \{1, 2\}$. Note that f_1 is

an open disk, hence $\text{surf}(f_1) = 0$. Using (4), we get

$$\begin{aligned} w(H, \mathcal{Q}) &= w(f) - w(f_1) - w(f_2) + w(H_1, \mathcal{Q}_1) + w(H_2, \mathcal{Q}_2) \\ &\leq w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) + \\ &\quad + \eta \sum_{i=1}^2 \text{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) + \\ &\quad + \eta \left(\text{surf}(f) - \text{surf}(f_2) - \sum_{h \in F(H)} \text{surf}(h) \right). \end{aligned}$$

Note that $w(f) - w(f_1) - w(f_2) \leq 16$ and $\ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) = \ell(\mathcal{Q})$. Also, $\text{surf}(f) - \text{surf}(f_2) \leq 48$, and when $g(\Pi_f) = 0$ and f has only two facial walks, then $\text{surf}(f) - \text{surf}(f_2) \leq 6$.

By Lemma 5.4(c), we have

$$\sum_{i=1}^2 \text{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) \leq \text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - \delta,$$

where $\delta = 16$ if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$ and $\delta = 56$ otherwise. Note that if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$, then $g(\Pi_f) = 0$ and f has only two facial walks. We conclude that $\text{surf}(f) - \text{surf}(f_2) - \delta \leq -8$. Therefore,

$$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta \left(\text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 8 - \sum_{h \in F(H)} \text{surf}(h) \right).$$

- *c is not separating.* Let f_1 be the face of H (in the embedding in Π_1) bounded by W and f_2 the other face corresponding to f . Again, note that $\text{surf}(f_1) = 0$. By (4) applied to the embedding of H in Π_1 , we obtain the following for the weight of H in Π :

$$\begin{aligned} w(H, \mathcal{Q}) &\leq w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}) + \\ &\quad + \eta \cdot \text{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \\ &\quad + \eta \left(\text{surf}(f) - \text{surf}(f_2) - \sum_{h \in F(H)} \text{surf}(h) \right). \end{aligned}$$

Since c is two-sided, $g(\Pi_1) = g(\Pi) - 2$, and

$$\text{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 124$$

by Lemma 5.4(d). Since $\text{surf}(f) - \text{surf}(f_2) \leq 48$ and $w(f) - w(f_1) - w(f_2) \leq 16$, we have

$$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta \left(\text{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 76 - \sum_{h \in F(H)} \text{surf}(h) \right).$$

The results of all the subcases imply (3). \square

Next, we consider the case that one of the faces is omnipresent.

Let H be an internally triangle-free graph embedded in Σ with rings \mathcal{R} and without non-contractible 4-cycles, and let f be an omnipresent face of H . If H is \mathcal{R} -critical, then

$$w(H, \mathcal{R}) \leq \ell(\mathcal{R}) - 2 = \ell(\mathcal{R}) - 2 + \eta \left(\text{surf}(g, |\mathcal{R}|, t_0, t_1) - \sum_{h \in F(H)} \text{surf}(h) \right). \quad (5)$$

Proof. Since H is \mathcal{R} -critical and f is an omnipresent face, each component of H is planar and contains exactly one ring. In particular, all faces of H distinct from f are closed 2-cell. For $R \in \mathcal{R}$, let H_R be the component of H containing R . Exactly one boundary walk W of f belongs to H_R . Cutting along W and capping the hole by a disk, we obtain an embedding of H_R in a disk with one ring R . Let f_R be the face of this embedding bounded by W . Note that either $H_R = R$ or H_R is R -critical. If R is a vertex-like ring, then we have $H_R = R$ by Theorem 3.7; hence, every vertex-like ring in \mathcal{R} forms a facial walk of f , and thus $\text{surf}(f) = \text{surf}(g, |\mathcal{R}|, t_0, t_1)$. Consequently, $\text{surf}(g, |\mathcal{R}|, t_0, t_1) = \sum_{h \in F(H)} \text{surf}(h)$, and it suffices to prove the first inequality of the claim.

Suppose that $H_R \neq R$ for a ring $R \in \mathcal{R}$. By Corollary 5.3, we have $w(H_R, \{R\}) \leq s(|R| - 2)$. Since f_R is a face of H_R , it follows that $|f_R| \leq |R| - 2$. Furthermore, $w(H_R, \{R\}) - w(f_R) \leq s(|R| - 2) - s(|f_R|) \leq |R| - |f_R| - 2$. On the other hand, if $H_R = R$, then $w(H_R, \{R\}) - w(f_R) = 0 = |R| - |f_R|$. As H is \mathcal{R} -critical, there exists $R \in \mathcal{R}$ such that $H_R \neq R$, and thus

$$\begin{aligned} w(H, \mathcal{R}) &= w(f) + \sum_{R \in \mathcal{R}} (w(H_R, \{R\}) - w(f_R)) \\ &\leq |f| + \left(\sum_{R \in \mathcal{R}} |R| - |f_R| \right) - 2 \\ &= \ell(\mathcal{R}) - 2. \end{aligned}$$

□

Finally, consider the case that the embedding is open 2-cell but not semi-closed 2-cell.

Let H be an internally triangle-free \mathcal{R} -critical graph embedded in Σ with rings \mathcal{R} (the same surface and rings as G) so that all faces of H are open 2-cell and without non-contractible 4-cycles. If $|E(H)| \leq |E(G)|$ and a face f of H is not semi-closed 2-cell, then

$$w(H, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \left(\text{surf}(g, |\mathcal{R}|, t_0, t_1) - 1/2 \right). \quad (6)$$

Proof. Since f is not semi-closed 2-cell, there exists a vertex v appearing at least twice in the facial walk of f , and a simple closed curve c going through the interior of f and joining two of the appearances of v ; furthermore, if v is the main vertex of a vertex-like ring, it is not the case that c separates R from the rest of the graph H . Cut the surface along c and patch the resulting hole(s) by disk(s). Let v_1 and v_2 be the two vertices to that v is split. For $i = 1, 2$, if v_i is not incident with a cuff in the resulting surface, drill a new hole next to it in the incident patch and add a triangle T_i tracing its boundary, with vertex set consisting of v_i and two new vertices (note that if v is an internal vertex, a hole will be drilled at both v_1 and v_2 , while if v is a ring vertex, a hole will be drilled at exactly one of v_1 and v_2).

If the curve c is separating, then let H_1 and H_2 be the resulting graphs embedded in the two surfaces Σ_1 and Σ_2 obtained by this construction; if c is not separating, then let H_1 be the resulting graph embedded in a surface Σ_1 . We choose the labels so that $v_1 \in V(H_1)$. If c is two-sided, then let f_1 and f_2 be the faces to that f is split by c , where f_1 is a face of H_1 . If c is one-sided, then let f_1 be the face in Σ_1 corresponding to f . Note that $|f_1| + |f_2| > |f|$ in the former case, and thus $w(f) - w(f_1) - w(f_2) < 16$. In the latter case, we have $w(f) = w(f_1)$.

If c is separating, then for $i \in \{1, 2\}$, let \mathcal{R}_i consist of the rings of $\mathcal{R} \setminus \{v\}$ contained in Σ_i , and if none of these rings contains v_i (so that T_i exists), then also of the vertex-like ring T_i . Here, we designate T_i as weak if v is an internal vertex, Σ_{3-i} is a cylinder and the ring of H_{3-i} distinct from T_{3-i} is a vertex-like ring. If c is not separating, then let \mathcal{R}_1 consist of the rings of \mathcal{R} , together with those of T_1 and T_2 that exist. In this case, we treat T_1 and T_2 as non-weak vertex-like rings.

Suppose first that c is not separating. By Lemma 3.8, H_1 is \mathcal{R}_1 -critical. Note that H_1 has at most two more (non-weak vertex-like) rings than H and $g(\Sigma_1) \in \{g-1, g-2\}$ (depending on whether c is one-sided or not), and that H_1 has at least two rings. If H_1 has only one more ring than H , then

$$\begin{aligned} \text{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \text{surf}(g-1, |\mathcal{R}|+1, t_0, t_1+1) \\ &\leq \text{gen}(g-1, |\mathcal{R}|+1, t_0, t_1+1) + 32 \\ &= \text{gen}(g, |\mathcal{R}|, t_0, t_1) - 44 \\ &= \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 44. \end{aligned}$$

Let us now consider the case that H_1 has two more rings than H (i.e., that v is an internal vertex). If $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$, then note that both rings of H_1 are vertex-like rings. Corollary 3.10 implies that H_1 has only one edge; but the corresponding edge in H would form a loop, which is a contradiction. Consequently, we have $g(\Sigma_1) > 1$ or $|\mathcal{R}_1| \geq 3$, and

$$\begin{aligned} \text{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \text{surf}(g-1, |\mathcal{R}|+2, t_0, t_1+2) \\ &= \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 32. \end{aligned}$$

By the induction hypothesis, we can apply Theorem 4.1 to H_1 , concluding that $w(H, \mathcal{R}) \leq \ell(\mathcal{R}) + 22 + \eta(\text{surf}(g, |\mathcal{R}|, t_0, t_1) - 32)$, and the claim follows.

Next, we consider the case that c is separating. Let us remark that H_i is \mathcal{R}_i -critical for $i \in \{1, 2\}$. This follows from Lemma 3.8 unless T_i is a weak vertex-like ring. If T_i is a weak vertex-like ring, then Σ_{3-i} is a cylinder and the ring R_{3-i} of H_{3-i} distinct from T_{3-i} is a vertex-like ring. By Corollary 3.10, H_{3-i} is a single edge and R_{3-i} is not weak. Consider any edge e of H_i not belonging to the rings. Since H is \mathcal{R} -critical, there exists a precoloring ψ of \mathcal{R} that extends to $H - e$, but not to H . Let ψ_1 be obtained from ψ by giving a non-main vertex of T_i the color of the main vertex of R_{3-i} . Then clearly ψ_1 extends to $H_i - e$, but not to H . This implies that H_i is indeed \mathcal{R}_i -critical.

Thus, we can apply Theorem 4.1 inductively for H_1 and H_2 , and we have

$$\begin{aligned} w(H, \mathcal{R}) &= w(H_1, \mathcal{R}_1) + w(H_2, \mathcal{R}_2) + w(f) - w(f_1) - w(f_2) \\ &\leq \ell(\mathcal{R}) + 22 + \eta \sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \end{aligned}$$

Therefore, since $\eta \geq 44$, to prove (6) it suffices to prove that

$$\sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 1. \quad (7)$$

Let us first consider the case that say $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 1$; i.e., c passes through a vertex incident with a ring R of H and surrounds this ring. By Lemma 3.2, the graph H_1 is 3-connected, and in particular by the choice of c , we conclude the ring R is not vertex-like. Hence, $\text{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\Sigma_1), t_1(\Sigma_1)) = 0$, $g(\Sigma_2) = g(\Sigma)$, $|\mathcal{R}_2| = |\mathcal{R}|$, $t_0(\mathcal{R}_2) = t_0(\mathcal{R})$ and $t_1(\mathcal{R}_2) = t_1(\mathcal{R}) + 1$. This implies

$$\sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - \delta,$$

where $\delta = 8$ if $g(\Sigma_2) = 0$, $|\mathcal{R}_2| = 2$, and both rings of \mathcal{R}_2 are vertex-like, and $\delta = 4$ otherwise.

Therefore, we can assume that for $i \in \{1, 2\}$, if $g(\Sigma_i) = 0$, then $|\mathcal{R}_i| \geq 2$. If $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 1$, we have

$$\begin{aligned} \sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &\leq \sum_{i=1}^2 (\text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 32) \\ &= \text{gen}(g, |\mathcal{R}|, t_0, t_1) - 12 \\ &= \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 12. \end{aligned}$$

Therefore, we can assume that $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 2$, i.e., v is an internal vertex. If for both $i \in \{1, 2\}$, we have $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| > 2$, then $g(\Sigma_1) +$

$g(\Sigma_2) = g(\Sigma)$, $t_0(\mathcal{R}_1) + t_0(\mathcal{R}_2) = t_0(\mathcal{R})$, $t_1(\mathcal{R}_1) + t_1(\mathcal{R}_2) = t_1(\mathcal{R}) + 2$, and

$$\begin{aligned} \sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &= \sum_{i=1}^2 \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\ &= \text{gen}(g, |\mathcal{R}|, t_0, t_1) - 32 \\ &= \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 32. \end{aligned}$$

and the claim follows.

Hence, we can assume that say $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$. Then, $\mathcal{R}_1 = \{T_1, R_1\}$ for some ring R_1 , $g(\Sigma_2) = g$ and $|\mathcal{R}_2| = |\mathcal{R}|$. Since H_1 is \mathcal{R}_1 -critical, Corollary 3.10 implies that R_1 is not a weak vertex-like ring. If R_1 is a vertex-like ring, then T_2 is a weak vertex-like ring of \mathcal{R}_2 which replaces the non-weak vertex-like ring R_1 . Therefore, $\text{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 1$. Furthermore, $\text{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \text{surf}(0, 2, 0, 2) = 0$, and the claim follows.

Finally, consider the case that R_1 is a non-vertex-like ring. By symmetry, we can assume that if $g(\Sigma_2) = 0$ and $|\mathcal{R}_2| = 2$, then \mathcal{R}_2 also contains a non-vertex-like ring. Since \mathcal{R}_2 is obtained from \mathcal{R} by replacing R_1 by a non-weak vertex-like ring T_2 , we have $\text{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 4$. Furthermore, $\text{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \text{surf}(0, 2, 0, 1) = 2$. Consequently,

$$\sum_{i=1}^2 \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 2.$$

Therefore, the inequality (7) holds. This proves (6). \square

By (3), (5) and (6), we either have $w(G, \mathcal{R}) < \eta \cdot \text{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R})$ or every face of G is semi-closed 2-cell. From now on, assume that the latter holds.

Suppose that there exists a path $P \subset G$ of length at most 12 with ends in distinct rings $R_1, R_2 \in \mathcal{R}$. By choosing the shortest such path, we can assume that P intersects no other rings. Let $J = P \cup \bigcup_{R \in \mathcal{R}} R$ and let $S = \{f\}$, where f is the unique face of J . Let $\{G'\}$ be the G -expansion of S , let Σ' be the surface in that G' is embedded and let \mathcal{R}' be the natural rings of G' . Note that $g(\Sigma') = g$, $|\mathcal{R}'| = |\mathcal{R}| - 1$, $\ell(\mathcal{R}') \leq \ell(\mathcal{R}) + 24$ and $t_0(\mathcal{R}') + t_1(\mathcal{R}') \geq t_0 + t_1 - 2$. Since $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$, by induction and by Lemma 5.4(a) we have $w(G, \mathcal{R}) = w(G', \mathcal{R}') \leq \eta \cdot \text{surf}(g, |\mathcal{R}'|, t_0(\mathcal{R}'), t_1(\mathcal{R}')) + \ell(\mathcal{R}') + 24 < \eta \cdot \text{surf}(g, |\mathcal{R}|, t_0, t_1) + \ell(\mathcal{R})$. Therefore, we can assume that no such path exists in G .

Let $R \in \mathcal{R}$ be a ring of G . Suppose G contains a subgraph P with at most 12 edges such that P is

- a path joining two distinct vertices u and v of R and otherwise disjoint from R such that none of the cycles in $R \cup P$ is contractible, or
- a non-contractible cycle not surrounding R , containing a vertex $u = v$ of R and otherwise disjoint from R , or

- a non-contractible cycle not surrounding R and disjoint from R , together with a path joining one of its vertices to a vertex $u = v$ of R .

Note that the existence of P implies that either $g > 0$ or $|\mathcal{R}| \geq 3$, and thus $\text{surf}(g, |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) = \text{gen}(g, |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R}))$. Since the distance between any two rings in G is at least 13, all vertices of $V(P) \setminus \{u, v\}$ are internal. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of all faces of J . Let $\{G_1, \dots, G_k\}$ be the G -expansion of S , and for $1 \leq i \leq k$, let Σ_i be the surface in that G_i is embedded and let \mathcal{R}_i be the natural rings of G_i . Note that $\sum_{i=1}^k t_0(\mathcal{R}_i) = t_0 - b_0$ and $\sum_{i=1}^k t_1(\mathcal{R}_i) = t_1 - b_1$, where $b_0 = 1$ if R is a weak vertex-like ring and $b_0 = 0$ otherwise, and $b_1 = 1$ if R is a non-weak vertex-like ring and $b_1 = 0$ otherwise. Let $r = \left(\sum_{i=1}^k |\mathcal{R}_i|\right) - |\mathcal{R}|$ and observe that either $r = 0$ and $k = 1$, or $r = 1$ and $1 \leq k \leq 2$ (depending on whether the curves in $\hat{\Sigma}$ corresponding to cycles in $R \cup P$ distinct from R are one-sided, two-sided and non-separating or two-sided and separating). Furthermore, as seen from this case analysis, $\sum_{i=1}^k g(\Sigma_i) = g + 2k - r - 3$.

We claim that $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k$. This is clearly the case, unless $g(\Sigma_i) = g$. Then, we have $k = 2$, $r = 1$ and $g(\Sigma_{3-i}) = 0$. Since the boundaries of faces of $R \cup P$ are non-contractible, Σ_{3-i} is not a disk, hence $|\mathcal{R}_{3-i}| \geq 2$ and $|\mathcal{R}_i| < |\mathcal{R}|$, again implying $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$.

By induction, we have $w(G_i, \mathcal{R}_i) \leq \ell(\mathcal{R}_i) + \eta \cdot \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$, for $1 \leq i \leq k$. Since every face of G corresponds to a face of G_i for some $i \in \{1, \dots, k\}$ and $\sum_{i=1}^k \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 24$, we conclude that

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 24 + \eta \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)).$$

Note that for $1 \leq i \leq k$, we have that Σ_i is not a disk and \mathcal{R}_i contains at least one non-vertex-like ring, and thus $\text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30$. Therefore,

$$\begin{aligned} & \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\ & \leq \sum_{i=1}^k (\text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30) \\ & = \text{gen}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 3) + 48r - 120(k - 1) + 30k + 4b_1 + 5b_0 \\ & = \text{gen}(g, |\mathcal{R}|, t_0, t_1) + 150k - 72r - 240 + 4b_1 + 5b_0 \\ & \leq \text{gen}(g, |\mathcal{R}|, t_0, t_1) - 7 = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 7. \end{aligned}$$

The inequality of Theorem 4.1 follows. Therefore, we can assume that G contains no such subgraph P .

Suppose that G contains a connected essential subgraph H with at most 12 edges (implying that $g > 0$ or $|\mathcal{R}| \geq 3$, and that $\text{surf}(g, |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) =$

$\text{gen}(g, |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R}))$). We can assume that H is minimal, i.e., no proper connected subgraph of H is essential. Given the already excluded cases, all vertices of H are internal and H satisfies one of the following conditions (see [13]):

- (a) H is a non-contractible cycle not surrounding any of the rings, or
- (b) H is the union of two cycles C_1 and C_2 intersecting in exactly one vertex, and C_1 and C_2 surround distinct rings, or
- (c) H is the union of two vertex-disjoint cycles C_1 and C_2 and of a path between them, and C_1 and C_2 surround distinct rings, or
- (d) H is the theta-graph and each of the three cycles in G surrounds a different ring (in this case, Σ is the sphere with three holes).

Let J be the subgraph of G consisting of H and of the union of the rings, and let S be the set of all faces of J . Let $\{G_1, \dots, G_k\}$ be the G -expansion of S , and for $1 \leq i \leq k$, let Σ_i be the surface in that G_i is embedded and let \mathcal{R}_i be the natural rings of G_i . Let $r = \left(\sum_{i=1}^k |\mathcal{R}_i|\right) - |\mathcal{R}|$. Note that $\sum_{i=1}^k t_0(\mathcal{R}_i) = t_0$, $\sum_{i=1}^k t_1(\mathcal{R}_i) = t_1$ and $\sum_{i=1}^k \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 24$.

Let us first consider the case (a). Observe that we have either $r = 1$ and $k = 1$, or $r = 2$ and $1 \leq k \leq 2$, and that $\sum_{i=1}^k g(\Sigma_i) = g - r + 2k - 2$. If $g(\Sigma_1) = g$, then $k = 2$ and $g(\Sigma_2) = 0$; furthermore, Σ_2 has at least three cuffs, since H does not surround a ring. Thus, if $g(\Sigma_1) = g$, then $r = 2$ and $|\mathcal{R}_1| = |\mathcal{R}| + r - |\mathcal{R}_2| < |\mathcal{R}|$. The same argument can be applied to Σ_2 if $k = 2$, hence $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k$.

By induction, we conclude that

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 24 + \eta \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)). \quad (8)$$

For $1 \leq i \leq k$, let $\delta_i = 72$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 1$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 2$, and let $\delta_i = 0$ otherwise, and note that since \mathcal{R}_i contains a non-vertex-like ring, we have

$$\text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) = \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i.$$

If $k = 2$, then recall that since H does not surround a ring, we have either $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| \geq 3$ for $i \in \{1, 2\}$; hence, $\delta_1 + \delta_2 = 0$. If $k = 1$, then recall that we can assume that G is not embedded in the projective plane with no rings by Theorem 1.2; hence, either $g(\Sigma_1) > 0$, or $|\mathcal{R}_1| \geq 2$. Consequently, we have $\delta_1 \leq 30$.

Combining the inequalities, we obtain $\sum_{i=1}^k \delta_i \leq 60 - 30k$, and

$$\begin{aligned}
& \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&= \sum_{i=1}^k \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i \\
&\leq \text{gen}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 2) + 48r - 120(k - 1) + \sum_{i=1}^k \delta_i \\
&\leq \text{gen}(g, |\mathcal{R}|, t_0, t_1) + 90k - 72r - 60 \\
&\leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 24.
\end{aligned}$$

Together with (8), this implies the inequality of Theorem 4.1.

If H satisfies (b), (c) or (d), then $k = 3$, $r = 3$ and say Σ_1 and Σ_2 are cylinders and Σ_3 is obtained from Σ by replacing two cuffs by one. Therefore, we can again apply the induction hypothesis to obtain (8). Furthermore, denoting $b_{i,j} = t_j(\mathcal{R}_i)$ (so $0 \leq b_{i,0} + b_{i,1} \leq 1$) for $i \in \{1, 2\}$ and $j \in \{0, 1\}$,

$$\begin{aligned}
& \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&= \text{surf}(g, |\mathcal{R}| - 1, t_0 - b_{1,0} - b_{2,0}, t_1 - b_{1,1} - b_{2,1}) + \sum_{i=1}^2 \text{surf}(0, 2, b_{i,0}, b_{i,1}) \\
&\leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 6.
\end{aligned}$$

This again implies the inequality of Theorem 4.1. Therefore, assume that

every connected essential subgraph of G has at least 13 edges. (9)

Suppose now G contains a path P of length one or two intersecting $\bigcup \mathcal{R}$ exactly in its endpoints u and v . Both ends of P belong to the same ring R ; let P , P_1 and P_2 be the paths in $R \cup P$ joining u and v . Recall that we can assume that $P \cup P_2$ is a contractible cycle. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of faces of J . Let $\{G_1, G_2\}$ be the G -expansion of S , and for $1 \leq i \leq 2$, let Σ_i be the surface in that G_i is embedded and let \mathcal{R}_i be the natural rings of G_i . Note that say Σ_2 is a disk and its ring has length $|P| + |P_2|$, and Σ_1 is homeomorphic to Σ . Therefore, by induction and Corollary 5.3, we have $w(G, \mathcal{R}) = w(G_1, \mathcal{R}_1) + w(G_2, \mathcal{R}_2) \leq \ell(\mathcal{R}_1) + \text{surf}(g, |\mathcal{R}|, t_0, t_1) + s(\ell(\mathcal{R}_2))$. Furthermore, $\ell(\mathcal{R}_1) + s(\ell(\mathcal{R}_2)) \leq \ell(\mathcal{R}_1) + \ell(\mathcal{R}_2) - 4 = \ell(R) + 2|P| - 4 \leq \ell(R)$, and the claim of the theorem follows. Therefore, we can assume that

G contains no path of length one or two intersecting \mathcal{R} exactly in its endpoints. (10)

Note that every 4-cycle in G is contractible, and thus it bounds a face by Lemma 3.6 and Theorem 3.7. Let us consider the case that every 4-face in G is ring-bound. Since the distance between rings is at least 13, by (10) this implies that every 4-face is incident with the main vertex of a vertex-like ring. For each vertex-like ring $R \in \mathcal{R}$ whose main vertex r is incident with a 4-face, choose one such 4-face bounded by a cycle $rw_2w_3w_4$ and let J_R be the edge rw_2 together with the (≤ 6) -cycle surrounding R and containing the path $w_2w_3w_4$. For any other ring $R \in \mathcal{R}$, let $J_R = \emptyset$. Since the distance between rings is at least 13, the subgraphs J_R for $R \in \mathcal{R}$ are pairwise vertex-disjoint. Let $J = \bigcup_{R \in \mathcal{R}} R \cup J_R$ and let S be the set of all faces of J . Let $\{G_1, \dots, G_k\}$ be the G -expansion of S , and for $1 \leq i \leq k$, let Σ_i be the surface in that G_i is embedded and let \mathcal{R}_i be the natural rings of G_i , labelled so that Σ_1 is homeomorphic to Σ and $\Sigma_2, \dots, \Sigma_k$ are disks bounded by rings of length at most 11, corresponding to the vertex-like rings of G incident with 4-faces. Note that G_1 has internal girth at least five, and thus $w(G_1, \mathcal{R}_1) \leq \ell(R_1) + \eta_0 \cdot \text{surf}(g, |\mathcal{R}|, 0, 0)$ by Theorem 3.4. Furthermore, $w(G_i, \mathcal{R}_i) \leq s(9)$ for $2 \leq i \leq k$ by Corollary 5.3. Therefore,

$$\begin{aligned} w(G, \mathcal{R}) &= \sum_{i=1}^k w(G_i, \mathcal{R}_i) \\ &\leq \ell(\mathcal{R}_1) + \eta_0 \cdot \text{surf}(g, |\mathcal{R}|, 0, 0) + |\mathcal{R}|s(9) \\ &\leq \ell(\mathcal{R}) + \eta_0 \cdot \text{surf}(g, |\mathcal{R}|, 0, 0) + |\mathcal{R}|(6 + s(9)) \\ &\leq \ell(\mathcal{R}) + \eta \cdot \text{surf}(g, |\mathcal{R}|, t_0, t_1). \end{aligned}$$

Finally, suppose that G contains a 4-face which is not ring-bound. Let G' be the \mathcal{R} -critical graph embedded in Σ such that $|E(G')| < |E(G)|$, and let $J \subseteq G$ and $\{S_f : f \in F(G')\}$ be the cover of G by faces of G' , obtained by Lemma 5.1. If G' does not contain any non-contractible 4-cycle, then let $G_0 = G'$ and $\mathcal{R}_0 = \mathcal{R}$.

Otherwise, consider the unique non-contractible 4-cycle $C = w_1w_2w_3w_4$ in G' , which is flippable to a 4-face, and let $R \in \mathcal{R}$ be the ring surrounded by C . Let Δ be the closed disk bounded by C in $\Sigma + \hat{C}_R$, where C_R is the cuff corresponding to R . By symmetry, we can assume that a face f_1 whose boundary contains the path $w_1w_2w_3$ is a subset of Δ , and the face f_2 whose boundary contains the path $w_1w_4w_3$ is not a subset of Δ . See Figure 2, which also illustrates the transformation we describe in the following paragraph.

Note that the subgraph H of G' drawn in Δ is connected (since G' is critical, Theorem 1.1 implies that any component of H contains R or C , and if C were in a different component from R , then by Theorem 3.7 the cycle C would form a component of H by itself, and due to the face f_2 , w_4 would be an internal vertex of G' of degree two, contradicting Lemma 3.1). Hence, the face f_1 is open 2-cell. Let a denote the walk such that the boundary of f_1 is the concatenation of a and $w_1w_2w_3$. Note that w_1 and w_3 form a 2-cut in G' . Let G_0 be the embedding of G' in Σ obtained by mirroring $H - w_2$ (i.e., the cyclic orders of neighbors of vertices in $H - \{w_1, w_2, w_3\}$ are reversed, and the orders of the neighbors of w_1 and w_3 in $H - w_2$ are reversed). Let \mathcal{R}_0 be the rings of G_0 obtained from \mathcal{R}

by mirroring R . Note that G_0 contains no non-contractible 4-cycle, as C is the only 4-cycle in G' containing w_1 and w_3 and C becomes a 4-face in G_0 . Let f be the face of G_0 whose boundary contains the walk a . Note that $\Sigma_f = \Sigma_{f_2}$. We have $w(G', \mathcal{R}) = w(G_0, \mathcal{R}) - w(f) + w(f_1) + w(f_2)$. If f_2 is not open 2-cell, then $w(f) - w(f_1) - w(f_2) = |f| - s(|f_1|) - |f_2| = |f_1| - 4 - s(|f_1|) \geq 0$. If f_2 is open 2-cell, then $w(f) - w(f_1) - w(f_2) = s(|f|) - s(|f_1|) - s(|f_2|) = s(|f_1| + |f_2| - 4) - s(|f_1|) - s(|f_2|) \geq 0$. Therefore,

$$w(G', \mathcal{R}) \leq w(G_0, \mathcal{R}_0). \quad (11)$$

Note that (11) holds trivially when $G_0 = G'$.

For $f \in F(G')$, let $\{G_1^f, \dots, G_{k_f}^f\}$ be the G -expansion of S_f and for $1 \leq i \leq k_f$, let Σ_i^f be the surface in that G_i^f is embedded and let \mathcal{R}_i^f denote the natural rings of G_i^f . Since J and $\{S_f : f \in F(G')\}$ is a cover of G by faces of G' , we have

$$w(G, \mathcal{R}) = \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f). \quad (12)$$

Consider a face $f \in F(G')$. We have $g(\Sigma_f) \leq g$. If $g(\Sigma_f) = g$, then every component of G' is planar, and since G' is \mathcal{R} -critical and triangle-free, Theorem 1.1 implies that each component of G' contains at least one ring of \mathcal{R} ; consequently, f has at most $|\mathcal{R}|$ facial walks and Σ_f has at most $|\mathcal{R}|$ cuffs. Since the surfaces embedding the components of the G -expansion of S_f are fragments of Σ_f , we have $(G_i^f, \Sigma_i^f, \mathcal{R}_i^f) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k_f$: otherwise, we would have $m(G_i^f) = m(G)$, hence by the definition of G -expansion, the boundary of S_f would have to be equal to the union of rings in \mathcal{R} , contrary to the fact that $J \neq \bigcup \mathcal{R}$ by the definition of a cover of G by faces of G' .

Therefore, we can apply Theorem 4.1 inductively for G_i^f and we get

$$w(G_i^f, \mathcal{R}_i^f) \leq \ell(\mathcal{R}_i^f) + \eta \cdot \text{surf}(g(\Sigma_i^f), |\mathcal{R}_i^f|, t_0(\mathcal{R}_i^f), t_1(\mathcal{R}_i^f)). \quad (13)$$

Observe that since $\{\Sigma_1^f, \dots, \Sigma_{k_f}^f\}$ are fragments of Σ_f , we have

$$\sum_{i=1}^{k_f} \text{surf}(g(\Sigma_i^f), |\mathcal{R}_i^f|, t_0(\mathcal{R}_i^f), t_1(\mathcal{R}_i^f)) \leq \text{surf}(f),$$

and summing (13) for $i = 1, \dots, k_f$, we obtain

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \leq |f| + \text{el}(f) + \eta \cdot \text{surf}(f). \quad (14)$$

In case that f is semi-closed 2-cell, all fragments of f are disks and we can use Corollary 5.3 instead of Theorem 4.1 and strengthen the inequality (13) to

$$w(G_i^f, \mathcal{R}_i^f) \leq s(\ell(\mathcal{R}_i^f) - \delta_i),$$

where $\delta_i = 0$ if G_i^f is equal to its ring and $\delta_i = 2$ otherwise. Furthermore, Lemma 5.1 guarantees that $\text{el}(f) \leq 2$ and that if $\text{el}(f) > 0$, then S_f is non-trivial. Hence, summing these inequalities, we can strengthen (14) to

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \leq w(f). \quad (15)$$

As the sum of the elasticities of the faces of G' is guaranteed to be at most 4 by Lemma 5.1, inequalities (12) and (14) give

$$\begin{aligned} w(G, \mathcal{R}) &\leq \sum_{f \in F(G')} (w(f) + \text{el}(f) + \eta \cdot \text{surf}(f)) \\ &\leq w(G', \mathcal{R}) + 4 + \eta \sum_{f \in F(G')} \text{surf}(f). \end{aligned}$$

If G' has a face that is neither open 2-cell nor omnipresent, then (3) applied to G_0 and (11) imply that

$$w(G', \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \left(\text{surf}(g, |\mathcal{R}|, t_0, t_1) - 7 - \sum_{f \in F(G')} \text{surf}(f) \right),$$

and consequently G satisfies the outcome of Theorem 4.1. Therefore, we can assume that all faces of G' are either open 2-cell or omnipresent. Similarly, using (6) we can assume that if no face of G' is omnipresent, then all of them are semi-closed 2-cell.

Suppose first that G' has no omnipresent face. Using (12), (15), (11), and applying Theorem 4.1 inductively for G_0 , we have

$$\begin{aligned} w(G, \mathcal{R}) &= \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \\ &\leq \sum_{f \in F(G')} w(f) \\ &= w(G', \mathcal{R}) \leq w(G_0, \mathcal{R}_0) \\ &\leq \ell(\mathcal{R}) + \eta \cdot \text{surf}(g, |\mathcal{R}|, t_0, t_1), \end{aligned}$$

hence G satisfies the outcome of Theorem 4.1.

It remains to consider the case that G' has an omnipresent face h . Then, every component of G is a plane graph with one ring, and by Lemma 3.2, we conclude that every face of G different from h is closed 2-cell. Lemma 5.1 guarantees $\text{el}(h) \leq 2$. By (12), (14), (15) and (11), we have

$$\begin{aligned} w(G, \mathcal{R}) &\leq w(h) + \text{el}(h) + \eta \cdot \text{surf}(h) + \sum_{f \in F(G'), f \neq h} w(f) \\ &= w(G', \mathcal{R}) + \text{el}(h) + \eta \cdot \text{surf}(h) \leq w(G_0, \mathcal{R}_0) + 2 + \eta \cdot \text{surf}(h). \end{aligned}$$

However, by (5), we have $w(G_0, \mathcal{R}_0) \leq \ell(\mathcal{R}_0) - 2 = \ell(\mathcal{R}) - 2 + \eta(\text{surf}(g, |\mathcal{R}|, t_0, t_1) - \text{surf}(h))$, and the claim of the theorem follows. \square

For use in a future paper of this series, let us formulate a corollary of Theorem 4.1. Let G be a graph embedded normally in a surface Σ . For a real number η and a face f of G , let $w_\eta(f) = w(f) + \eta \cdot \text{surf}(f)$. We define $w_\eta(G)$ as the sum of $w_\eta(f)$ over the faces f of G .

Corollary 5.5. *There exists a constant $\eta > 0$ such that the following holds. Let G be a triangle-free graph embedded in a surface Σ without non-contractible 4-cycles and with rings \mathcal{R} . Let B be the union of the rings \mathcal{R} . If G is \mathcal{R} -critical, then $w_\eta(G) \leq w_\eta(B)$.*

Proof. Choose η as the constant of Theorem 4.1. Suppose first that G is embedded in the disk with one ring of length $l \geq 4$. Since G is \mathcal{R} -critical, it is connected and thus all faces of G are open 2-cell, and $w_\eta(f) = w(f)$ for every face f of G . Note furthermore that $w_\eta(B) = s(l)$. Hence, $w_\eta(G) = w(G, \mathcal{R}) \leq s(l - 2) \leq w_\eta(B)$ by Corollary 5.3.

Hence, we can assume that Σ is not the disk, and thus

$$w_\eta(B) = \ell(\mathcal{R}) + \eta \cdot \text{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})).$$

If all faces of the embedding of G in Σ are open 2-cell, then $w_\eta(G) = w(G, \mathcal{R})$ and the claim follows from Theorem 4.1. Otherwise, the claim follows from (3) or (5). \square

6 Forcing 3-colorability by removing $O(g)$ vertices

Finally, we show that in a triangle-free graph of genus g , it suffices to remove $O(g)$ vertices to make it 3-colorable.

Proof of Theorem 1.4. Let κ be the constant of Theorem 1.3. We let $\beta = \max(5\kappa, 4)$. We prove Theorem 1.4 by induction on the number of vertices. For $g = 0$, the claim holds by Grötzsch's theorem, hence assume that $g > 0$. Let G be a triangle-free graph embedded in a surface Σ of Euler genus g , such that Theorem 1.4 holds for all graphs with less than $|V(G)|$ vertices. Without loss of generality, we can assume that all the faces in the embedding of G are open 2-cell.

Suppose that G contains a non-facial 4-cycle K . If K is non-contractible, then $G - V(K)$ can be embedded in a surface of Euler genus at most $g - 1$. By the induction hypothesis, there exists a set X' of size at most $\beta(g - 1)$ such that $G - V(K) - X'$ is 3-colorable, and we can set $X = V(K) \cup X'$.

Hence, assume that K is contractible. Let G' be obtained from G by removing vertices and edges contained in the open disk bounded by K . By the induction hypothesis, there exists a set X of size at most βg such that $G' - X$ is 3-colorable. We claim that $G - X$ is also 3-colorable. Indeed, consider any 3-coloring φ of $G' - X$ and let H be the subgraph of G drawn in the closed disk bounded by K . Let ψ be a 3-coloring of K that matches φ on $V(K) \setminus X$. By

Theorem 3.7, ψ extends to a 3-coloring φ' of H . Hence, φ together with the restriction of φ' to $V(H) \setminus X$ gives a 3-coloring of $G - X$.

Therefore, we can assume that every 4-cycle in G bounds a face. If G is 4-critical, then let $X = \{v\}$ for an arbitrary vertex $v \in V(G)$. By the definition of a 4-critical graph, $G - X$ is 3-colorable. Hence, assume that G is not a 4-critical graph.

Let $G_0 \subseteq G$ be a maximal subgraph of G such that every component of G_0 is 4-critical. Let X consist of all vertices of G_0 that are incident with a face of length greater than 4. Firstly, we bound the size of X . Let H_1, \dots, H_k be the components of G_0 . For $i = 1, \dots, k$, let us obtain an embedding of H_i in a surface Σ_i such that every face is open 2-cell from the embedding of H_i in Σ as follows: as long as there exists a face f that is not open 2-cell, cut along a closed non-contractible curve contained in f and cap the resulting holes by disks. Since H_i is 4-critical, triangle-free and every 4-cycle in H_i bounds a face, Theorem 1.3 implies that $|X \cap V(H_i)| \leq 5\kappa g(\Sigma_i)$. Observe that since H_1, \dots, H_k are vertex-disjoint, we have $g \geq g(\Sigma_1) + \dots + g(\Sigma_k)$, and thus $|X| \leq 5\kappa g$ as required.

Consider any $i \in \{1, \dots, k\}$. If $V(H_i) \cap X = \emptyset$, then all faces of H_i have length 4. Since all 4-cycles in G bound faces, it follows that $G = H_i$. However, this contradicts the assumption that G is not 4-critical. Therefore, X intersects every component of G_0 .

We claim that $G - X$ is 3-colorable. Indeed, if that is not the case, then $G - X$ contains a 4-critical subgraph G' . By the maximality of G_0 , it follows that there exists $i \in \{1, \dots, k\}$ such that G' intersects H_i . Since $V(H_i) \cap X \neq \emptyset$, we have $G' \neq H_i$, and since G' and H_i are 4-critical, it follows that $G' \not\subseteq H_i$. Since G' is 4-critical, it is connected, and thus there exists $uv \in E(G)$ such that $u \in V(G') \setminus V(H_i)$ and $v \in V(G') \cap V(H_i)$. Let f be the face of H_i containing u , and note that v is incident with f . Since $v \in V(G')$, we have $v \notin X$, and thus f is a 4-face. However, then the 4-cycle bounding the face f separates u from all vertices of X in G , which contradicts the assumption that all 4-cycles in G bound faces. \square

References

- [1] AKSIONOV, V. A. On continuation of 3-colouring of planar graphs. *Diskret. Anal. Novosibirsk* 26 (1974), 3–19. In Russian.
- [2] DVOŘÁK, Z., KAWARABAYASHI, K., AND THOMAS, R. Three-coloring triangle-free planar graphs in linear time. *Trans. on Algorithms* 7 (2011), article no. 41.
- [3] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Coloring planar graphs with triangles far apart. *ArXiv 0911.0885v1* (Nov. 2009).

- [4] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces II. 4-critical graphs in a disk. *Journal of Combinatorial Theory, Series B* 132 (2018), 1–46.
- [5] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces III. Graphs of girth five. *Journal of Combinatorial Theory, Series B* 145 (2020), 376–432.
- [6] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies. *ArXiv* 0911.0885 (2020).
- [7] GIMBEL, J., AND THOMASSEN, C. Coloring graphs with fixed genus and girth. *Trans. Amer. Math. Soc.* 349 (1997), 4555–4564.
- [8] GRÖTZSCH, H. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Math.-Natur. Reihe* 8 (1959), 109–120.
- [9] HAVEL, I. On a conjecture of B. Grünbaum. *J. Combin. Theory, Ser. B* 7 (1969), 184–186.
- [10] HUTCHINSON, J. P. Three-coloring graphs embedded on surfaces with all faces even-sided. *J. Combin. Theory, Ser. B* 65 (1995), 139–155.
- [11] KAWARABAYASHI, K., AND THOMASSEN, C. From the plane to higher surfaces. *Journal of Combinatorial Theory, Series B* 102, 4 (2012), 852–868.
- [12] KOSTOCHKA, A. V., AND YANCEY, M. Ore's conjecture for $k = 4$ and grötzsch's theorem. *Combinatorica* 34, 3 (2014), 323–329.
- [13] ROBERTSON, N., AND SEYMOUR, P. D. Graph Minors VII. Disjoint paths on a surface. *J. Combin. Theory, Ser. B* 45 (1988), 212–254.
- [14] THOMAS, R., AND WALLS, B. Three-coloring Klein bottle graphs of girth five. *J. Combin. Theory, Ser. B* 92 (2004), 115–135.
- [15] THOMASSEN, C. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory, Ser. B* 62 (1994), 268–279.
- [16] THOMASSEN, C. 3-list-coloring planar graphs of girth 5. *J. Combin. Theory, Ser. B* 64 (1995), 101–107.
- [17] THOMASSEN, C. The chromatic number of a graph of girth 5 on a fixed surface. *J. Combin. Theory, Ser. B* 87 (2003), 38–71.
- [18] THOMASSEN, C. A short list color proof of Grötzsch's theorem. *J. Combin. Theory, Ser. B* 88 (2003), 189–192.